Test 3 Solutions

Calculus III **April 1, 2008**

1. Carefully sketch the region of integration for the integral $\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$. Then reverse the order of integration and evaluate the integral.

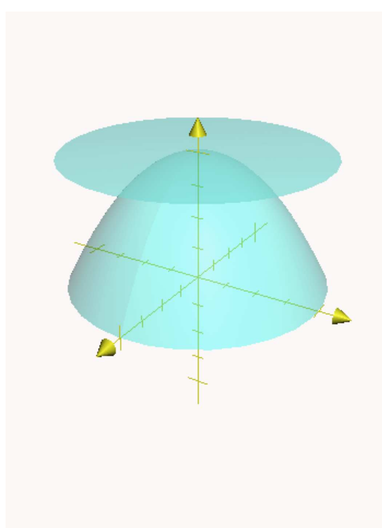
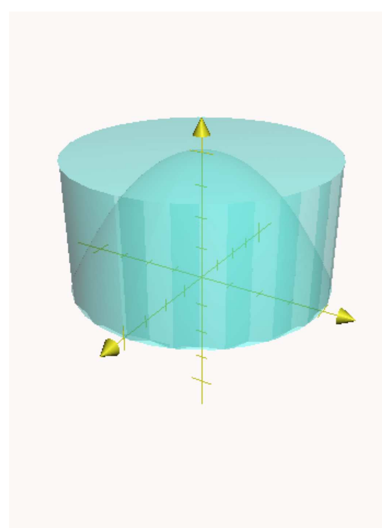
> The region lies between the lines $x = 3y$ and $x = 3$, at points where the y-coordinates lie between 0 and 1. This is the triangle shown below.
>
> Reversing the order of integration means treating this as a y-simple region, for which we get the bounds $0 \le y \le x/3$, $0 \le x \le 3$. Thus the new integral is

$$\int_0^3 \int_0^{x/3} e^{x^2}\,dy\,dx = \int_0^3 \left[ye^{x^2}\right]_{y=0}^{y=x/3} dx = \frac{1}{3}\int_0^3 \left[xe^{x^2}\right] dx$$

$$= \frac{1}{6}\int e^u\,du = \frac{1}{6}\,e^{x^2}\Big|_{x=0}^{x=3} = \frac{1}{6}(e^9 - 1).$$

2. A solid E lies within the vertical cylinder $x^2 + y^2 = 1$, below the plane $z = 1$, and above the paraboloid $z = 1 - x^2 - y^2$. Evaluate the integral

$$\iiint_E (x^2 + y^2)^{1/2}\,dV.$$

This region is z-simple, with $1 - x^2 - y^2 \le z \le 1$, so we have

$$\iiint_E (x^2+y^2)^{1/2}\,dV = \iint_D \int_{1-x^2-y^2}^{1} (x^2+y^2)^{1/2}\,dz\,dA.$$

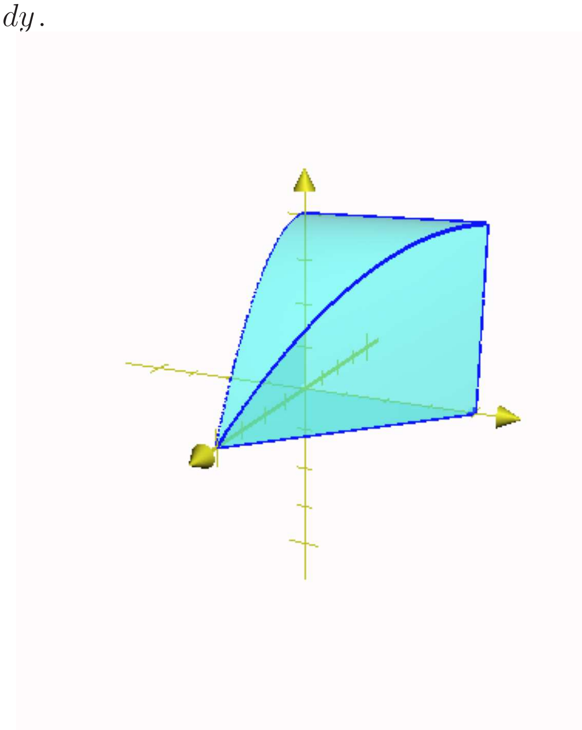
Now we handle the double integral. Note that D refers to points in the xy-plane, the vertical lines over which hit our region. Thus D is the region inside the unit circle. This region is most easily described in polar coordinates, and the function we're integrating looks nice with x and y switched to polar, also, so that's what we'll do. Thus we're ultimately using cylindrical coordinates for this integral. Now we have

$$\iiint_E (x^2+y^2)^{1/2}\,dV = \int_0^{2\pi}\int_0^1\int_{1-r^2}^{1} r\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left[r^2 z\right]_{z=1-r^2}^{z=1} dr\,d\theta = \int_0^{2\pi}\int_0^1 (r^2-(r^2-r^4))\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^4\,dr\,d\theta = \frac{2\pi}{5}.$$

3. Consider the region E shown below, bounded by the three coordinate planes, the plane $y = 1 - x$, and the surface $z = 1 - x^2$. Express the volume of E as a triple integral in two ways: $dy\,dz\,dx$ and $dz\,dx\,dy$.



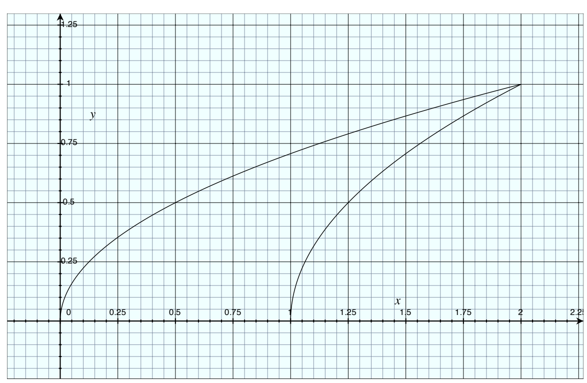
Lines parallel to the y-axis enter E on the surface $y = 0$ (the xz-plane) and exit the region on the surface $y = 1 - x$. The region of the xz-plane corresponding to those lines that actually hit E is the parabolic-roofed left-hand wall in the picture. As a z-simple region, it is described as $0 \le z \le 1 - x^2$, where $0 \le x \le 1$. Thus we have

$$\mathtt{Vol}(E) = \int_0^1 \int_0^{1-x^2} \int_0^{1-x} dy\, dz\, dx.$$

For the other order, note that lines parallel to the z-axis enter E on the plane $z = 0$ and exit E on the surface $z = 1 - x^2$. The region in the xy-plane corresponding to the lines that hit E is the triangular floor. As an x-simple region this is described as $0 \le x \le 1 - y$ with $0 \le y \le 1$. Thus we have

$$\mathtt{Vol}(E) = \int_0^1 \int_0^{1-y} \int_0^{1-x^2} dz\, dx\, dy.$$

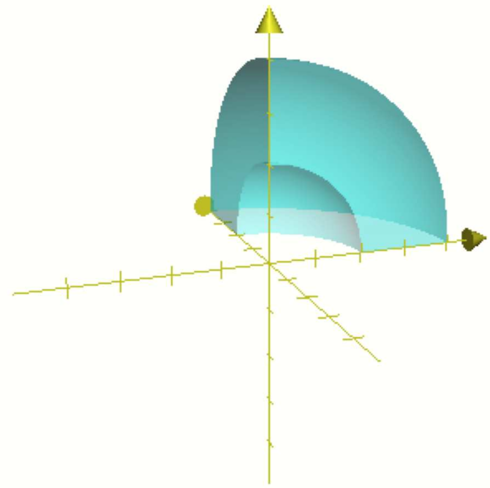
4. A thin metal sheet occupies the region D bounded by the parabolas $x = 2y^2$ and $x = 1 + y^2$ and lying above the x-axis. The density of the sheet at the point (x, y) is described by $d(x, y) = y + 2$. Find the mass of the sheet (i.e., integrate the density function over D).



This region is x-simple, where $2y^2 \le x \le 1 + y^2$ and $0 \le y \le 1$. Thus we have

$$\iint_D (y+2)\, dA = \int_0^1 \int_{2y^2}^{1+y^2} (y+2)\, dx\, dy = \int_0^1 (2 + y - 2y^2 - y^3)\, dy = \frac{19}{12}$$

5. Express $\iiint_E z\, dV$ as an iterated integral, where E is the region between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the first octant (recall that the *first octant* is the region where $x \ge 0$, $y \ge 0$, and $z \ge 0$).



We use spherical coordinates, in which the region is described by $1 \le \rho \le 2$, $0 \le \phi \le \pi/2$, $0 \le \theta \le \pi/2$. Thus we have

$$\iiint_E z\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho\cos\phi\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho^3\sin\phi\cos\phi\,d\phi\,d\theta.$$