Advanced Calculus II **Final Exam**

Due by 5pm April 30, 2008, in my office ILB 426

Name: OUID:

1. We define a measure n on $\mathbf{R}$ as follows: Let $n(E) = \infty$ if E is any infinite subset of $\mathbf{R}$; for a finite set E, let $n(E)$ be the number of elements in E. This is called the *counting measure* on $\mathbf{R}$. Show that n satisfies properties 1, 3, and 4.

2. Let $I = [a, b]$. Prove directly that $m(I) = b - a$, where m is Lebesgue measure.

3. Let E denote the rationals between zero and one. Show directly that $m(E) = 0$.

4. Show that any set E with $m^*(E) = 0$ is measurable.

5. Let C be the Cantor set. Show that C is uncountable and measurable with $m(C) = 0$. (You may assume that outer measure is *finitely* additive.)

> Hint: Recall that C can be thought of in two ways: (i) as those real numbers between zero and one having ternary expansions containing only 0s and 2s; and (ii) as those real numbers between zero and one that are left over after one repeatedly removes open middle thirds.

NOTES

The four ideal properties for a measure m on $\mathbf{R}$ are the following:

1. $m(E)$ is defined for all sets $E \subseteq \mathbf{R}$;
2. for all intervals $I = [a, b]$, $m(I) = b - a$;
3. (countable additivity) $m(\cup E_n) = \sum m(E_n)$ for any countable collection of pairwise disjoint sets $E_n \subseteq \mathbf{R}$;
4. (translation invariance) $m(E + x) = m(E)$ for all E and x, where $E + x = \{r \in \mathbf{R} \mid r - x \in E\}$.

No measure satisfies all four of these properties.

For any set $E \subseteq \mathbf{R}$, the *outer measure* of E is defined to be

$$m^*(E) = \inf \left\{ \sum \ell(I_n) \mid E \subseteq \cup I_n \right\},$$

where $I_n = (a_n, b_n)$. Outer measure satisfies properties 1, 2, and 4 above, but not 3.

A set E is called *measurable* if, for any other set A, we have $m^*(A) = m^*(A \cap E) + m^*(A - E)$. Lebesgue measure refers to outer measure restricted to the class of measurable sets. Lebesgue measure satisfies properties 2, 3, and 4 above, but not 1.