Homework 1 Solutions

Advanced Calculus II **January 23, 2008**

4.6 #2: Let f be a continuous real-valued function on a closed rectangle R in the plane. Then there exist points c and d in R so that $f(c) = \sup_{p \in R} f(p)$ and $f(d) = \inf_{p \in R} f(p)$.

We will prove the statement about c; the one about d is similar. Let $M = \sup_{p \in R} f(p)$. Then because M is the *least* upper bound, there is a sequence of points $p_1, p_2, \ldots$, each in R, so that $\lim_{n \to \infty} f(p_n) = M$. Now because R is sequentially compact, some subsequence of the p_n actually converges to some point, call it c, that is actually contained in R. Thus f is defined at c. Now because f is continuous, we must have $f(c) = f(\lim p_n) = \lim f(p_n) = M$.

4.6 #5: (a) Show that if $E \subseteq \mathbf{R}^2$ is closed and bounded, then E is sequentially compact.

Let p_n be a sequence of points in E. Because E is bounded, the Bolzano-Weierstrass theorem states that this sequence has an accumulation point, say p. In particular, some subsequence of this sequence must converge to p. Because E is closed, we know that p is in E. Thus, every sequence of points in E has a subsequence that converges to a point in E.

(b) Show that if E is not closed and bounded, then E is not sequentially compact.

If E is not closed, then there is some sequence of points in E converging to some $p \notin E$. This sequence shows that E is not sequentially compact. If E is not bounded, then there is some sequence of points in E diverging to infinity. This sequence also violates sequential compactness.

(c) Is $\{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \le 1\}$ compact?

Yes, as it is both closed and bounded.

(d) Is $\{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 < 1\}$ compact?

No, because it is not closed (it does not contain the limit point of the sequence $(1-1/n, 0)$, for example).

(e) Is $\{(x,y) \in \mathbf{R}^2 \mid x^2+y^2 \geq 1\}$ compact?

No, because it is unbounded.

4.6 #6: Give an example of a noncompact set $E \subseteq \mathbf{R}^2$ and a continuous function f on E so that (i) f is unbounded on E; (ii) f attains neither its inf nor its sup; and (iii) f is not uniformly continuous on E.

The function $f(x,y) = x^3$ will do, with $E = R^2$. It is clearly unbounded, attains either its inf $(-\infty)$ nor its sup (∞), and is not uniformly continuous (because it becomes arbitrarily steep as $|x|$ gets large).

4.6 #7: Show that the partial derivatives of $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$ exist at $(0,0)$ but that f is not differentiable there.

The function as defined in this problem is not defined at $(0,0)$. On the other hand, the definition of the partials at $(0,0)$ require that the function be defined there, so the problem is poorly stated. We will assume that $f(0,0) = 0$.

Now to show that the partials exist, it suffices to calculate them directly using the difference quotient. Thus for $\frac{\partial f}{\partial x}$ at $(0,0)$, we have

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{1}{h}\Big(f(h,0) - f(0,0)\Big) = \lim_{h \to 0} \frac{1}{h}(0-0) = 0,$$

and a similar calculation holds for $\frac{\partial f}{\partial y}$.

Recall that the book's definition of differentiable is that the partials exist and are continuous. So to show f is not differentiable at $(0,0)$, we need to show that the partials are not continuous there. For this, we calculate directly that

$$\frac{\partial f}{\partial x} = \frac{y^3}{(x^2+y^2)^{3/2}}.$$

This function is discontinuous at $(0,0)$, as can be seen by the following unequal limits: coming in to the origin along the x-axis (with $y=0$), we have

$$\lim_{x \to 0} \frac{0}{x^3} = 0;$$

coming in to the origin along the y-axis (with $x=0$), we have

$$\lim_{y \to 0} \frac{y^3}{y^3} = 1.$$