

## Homework 2 Solutions

### Advanced Calculus II

February 6, 2008

5.1 #2: Suppose that  $g$  is a continuous function on  $[a, b]$ .

(a) Prove that if  $f_n \rightarrow f$  pointwise, then  $gf_n \rightarrow gf$  pointwise.

Fix  $x \in [a, b]$  and let  $\epsilon > 0$  be given. Because  $f_n \rightarrow f$  pointwise, there is some  $N$  so that  $|f_n(x) - f(x)| < \frac{\epsilon}{|g(x)|}$  whenever  $n \geq N$  (the key point here is that  $g(x)$  is a fixed constant, since we've fixed  $x$ ). Then for  $n \geq N$  we have

$$\begin{aligned} |(gf_n)(x) - (gf)(x)| &= |g(x)f_n(x) - g(x)f(x)| \\ &= |g(x)||f_n(x) - f(x)| < |g(x)|\frac{\epsilon}{|g(x)|} = \epsilon, \end{aligned}$$

as required.

(b) Prove that if  $f_n \rightarrow f$  uniformly, then  $gf_n \rightarrow gf$  uniformly.

Let  $\epsilon > 0$  be given. Because  $g$  is continuous on the closed and bounded interval  $[a, b]$ , it has a finite supremum. Let  $M = \sup_{x \in [a, b]} \{g(x)\}$ . Now because  $f_n \rightarrow f$  uniformly, there is some  $N$  so that  $|f_n(x) - f(x)| < \frac{\epsilon}{M}$  whenever  $n \geq N$ , and this holds for all  $x \in [a, b]$ . Then for  $n \geq N$  we have

$$\begin{aligned} |(gf_n)(x) - (gf)(x)| &= |g(x)f_n(x) - g(x)f(x)| \\ &= |g(x)||f_n(x) - f(x)| < |g(x)|\frac{\epsilon}{M} \leq \epsilon, \end{aligned}$$

as required.

5.1 #5: Prove that  $f_n(x) = (x - \frac{1}{n})^2$  converges uniformly on any finite interval.

Fix an interval  $[a, b]$ . We claim that  $f_n$  converges uniformly to  $f(x) = x^2$  on  $[a, b]$ . Let  $\epsilon > 0$  be given, and let  $M = \max\{|a|, |b|, 1\}$ . Then we have

$$|f(x) - f_n(x)| = |x^2 - (x - \frac{1}{n})^2| = |\frac{2x}{n} - \frac{1}{n^2}| \leq \frac{2M}{n} + \frac{M}{n^2} \leq \frac{3M}{n}.$$

Thus if we take  $N = \frac{3M}{\epsilon}$ , then  $n \geq N$  implies that  $|f(x) - f_n(x)| < \epsilon$ .

**5.1 #7:** Let  $f_n(x) = \frac{nx}{1+n^2x^2}$ . Prove that  $f_n \rightarrow 0$  pointwise but not uniformly on  $[0, 1]$ .

Fix  $x$ . Then

$$\lim_{n \rightarrow \infty} \frac{nx}{1+n^2x^2} = \lim_{n \rightarrow \infty} \frac{x}{\frac{1}{n} + nx^2} = 0,$$

so that  $f_n \rightarrow 0$  pointwise.

On the other hand, note that  $f_n\left(\frac{1}{n}\right) = \frac{1}{2}$ , so that the convergence cannot be uniform (for any  $\epsilon < \frac{1}{2}$ , there will not be any  $n$  with  $f_n(x) < \epsilon$  at all  $x$ ).

**5.2 #1:** Let  $f_n(x) = e^{-nx}$  on the interval  $[0, 1]$ . Explain why the sequence of functions  $\{f_n\}$  converges pointwise on  $[0, 1]$ . What is the limiting function? Is it continuous? Is the convergence uniform?

Fix  $x \in [0, 1]$ . If  $x = 0$ , then  $f_n(x) = 1$  for all  $n$ , so that  $\lim_{n \rightarrow \infty} f_n(0) = 1$ . If  $x > 0$ , then  $\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} e^{-nx} = 0$ . Thus the limiting function is the function  $f(x)$  with  $f(0) = 1$  and  $f(x) = 0$  for all  $x \neq 0$ . This function is not continuous, so the convergence cannot be uniform on this interval.

**5.2 #2b:** Compute the limit:  $\lim_{n \rightarrow \infty} \int_1^2 e^{-nx} dx$ .

From the argument in the previous problem, we see that  $f_n(x) \rightarrow 0$  on  $[1, 2]$ . Moreover this convergence is uniform on this interval. (Fix  $\epsilon > 0$  and note that  $\sup_{x \in [1, 2]} \{f_n(x)\} = e^{-n}$ . Thus we have  $|f_n(x) - f(x)| = e^{-n}$ , so it suffices to choose  $N$  large enough so that  $e^{-N} < \epsilon$ .) So now we have

$$\lim_{n \rightarrow \infty} \int_1^2 e^{-nx} dx = \int_1^2 \lim_{n \rightarrow \infty} f_n(x) dx = \int_1^2 0 dx = 0.$$

**5.2 #3:** Give an example of a sequence of continuously differentiable functions  $\{f_n\}$  on  $[0, 1]$  so that  $f_n \rightarrow f$  uniformly but  $f$  is not differentiable at all points of  $[0, 1]$ .

A sequence of functions involving a curve that becomes sharper and sharper until in the limit it is a corner will do (think  $|x|$ ). Similarly a sequence of bounded functions whose derivatives are getting steeper and steeper will also do it (think  $\sqrt{x}$ ).

**5.3 #2:** Let  $f$  and  $g$  be continuous functions on  $[a, b]$ .

(a) Use the triangle inequality to prove that  $|\|f\|_\infty - \|g\|_\infty| \leq \|f - g\|_\infty$ .

Define  $h = f - g$ . Then by the triangle inequality, we have the following:

$$\|h + g\|_\infty \leq \|h\|_\infty + \|g\|_\infty.$$

Substituting back to  $f$  and  $g$  and rearranging gives

$$\|f\|_\infty \leq \|f - g\|_\infty + \|g\|_\infty \Leftrightarrow \|f - g\|_\infty \geq \|f\|_\infty - \|g\|_\infty.$$

Similarly, setting  $u = g - f$  and applying the triangle inequality to  $u$  and  $f$  gives

$$\|g - f\|_\infty \geq \|g\|_\infty - \|f\|_\infty.$$

The result now follows from the fact that

$$\|g - f\|_\infty = \|f - g\|_\infty.$$

(b) Suppose that  $f_n \rightarrow f$  in the sup norm. Prove that  $\|f_n\|_\infty \rightarrow \|f\|_\infty$ .

Let  $\epsilon > 0$  be given. Because  $f_n \rightarrow f$  in the sup norm, there is some  $N$  so that  $\|f_n - f\|_\infty < \epsilon$  whenever  $n \geq N$ . It follows then from part (a) that  $|\|f_n\|_\infty - \|f\|_\infty| < \epsilon$  whenever  $n \geq N$ . This is exactly the statement that  $\|f_n\|_\infty \rightarrow \|f\|_\infty$ .

**5.3 #3:** Let  $C_b(\mathbf{R})$  denote the set of bounded continuous functions on  $\mathbf{R}$ . Prove that  $C_b(\mathbf{R})$  is complete in the sup norm.

Suppose  $\{f_n\}$  is a sequence of continuous bounded functions on  $\mathbf{R}$  that is Cauchy in the sup norm, and let  $\epsilon > 0$  be given. Because the sequence is Cauchy, we know that there is some  $N$  so that  $\|f_n - f_m\|_\infty < \epsilon$  whenever  $n, m \geq N$ . Now fix  $x \in \mathbf{R}$ , and consider the sequence of real numbers  $\{f_n(x)\}$ . Note that for  $n, m \geq N$  we have  $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \epsilon$ , so that this sequence is Cauchy. Because  $\mathbf{R}$  is complete, it must converge. Thus we may define a function  $f: \mathbf{R} \rightarrow \mathbf{R}$  by  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ .

We claim that  $f_n(x) \rightarrow f(x)$  in the sup norm, and that  $f(x)$  is bounded and continuous.

To see that  $f$  is bounded, let  $B = \sup\{|f_N(x)|\}$ , where this is the same  $N$  as above. Then from the Cauchy property of this sequence of functions, we know that  $|f_n(x)| \leq B + \epsilon$  for all  $x$ . Then from the properties of limits, we see that

$f(x) = \lim_{n \rightarrow \infty} f_n(x) \leq B + \epsilon$  for all  $x$ . Thus  $f$  is bounded on  $\mathbf{R}$ .

To see that the convergence is uniform, note that for  $m, n \geq N$  we have for every  $x$  that

$$|f_n(x) - f(x)| = |f_n(x) - \lim_{m \rightarrow \infty} f_m(x)| = \lim_{m \rightarrow \infty} |f_n(x) - f_m(x)| < \epsilon,$$

(where we used continuity of the absolute value function to pull the limit out).

Finally, the fact that the limit is continuous follows from the fact that the convergence is uniform and each of the  $f_n$  is continuous.