Take-Home Test 1

Advanced Calculus II **Due February 20, 2008**

1. (Project 1, p. 219): You will show in this problem that if f is a continuous function on $[0,1]$, then $\| f \|_p \to \| f \|_\infty$ as $p \to \infty$.

(a) Show that for each p, $\| f \|_p \leq \| f \|_\infty$.

(b) Given that $f(x)$ is continuous on $[0,1]$, explain why $|f(x)|$ is also continuous on $[0,1]$. Then let x_0 be the point where $|f(x)|$ achieves its maximum. Explain why $|f(x_0)| = \| f \|_\infty$.

(c) Assume that x_0 is not one of the endpoints of $[0,1]$, and let $\epsilon > 0$ be given. Explain why you can choose a $\mu > 0$ so that $|f(x)| \geq \| f \|_\infty - \epsilon$ for all $x \in [x_0 - \mu, x_0 + \mu]$.

(d) Prove that $\int_0^1 |f(x)|^p \, dx \geq 2\mu(\| f \|_\infty - \epsilon)^p$ for all p.

(e) Show that for p large enough, $\| f \|_\infty \geq \| f \|_p \geq \| f \|_\infty - 2\epsilon$ and conclude that $\lim_{p \to \infty} \| f \|_p = \| f \|_\infty$.

2. In this problem we will show that the 2–norm satisfies the triangle inequality. Recall that the 2–norm is defined by

$$\| f \|_2 = \left(\int_a^b [f(x)]^2 \, dx \right)^{1/2}.$$

(a) Show that for any real number λ, we have

$$0 \leq \| f + \lambda g \|_2^2 = \| f \|_2^2 + 2\lambda \int fg + \lambda^2 \| g \|_2^2 .$$

(b) Use part (a) to prove the Cauchy-Schwarz inequality: $\left(\int fg \right)^2 \leq \| f \|_2^2 \| g \|_2^2$.
(Hint: Interpret the right-most expression in part (a) as a quadratic polynomial in λ. What does the fact that it is never negative tell you about its roots? What does this imply about the radical in the quadratic equation?)

(c) Prove the triangle inequality, $\| f + g \|_2 \leq \| f \|_2 + \| g \|_2$, by computing $\| f + g \|_2$ and applying the Cauchy-Schwarz inequality to the result.