Take-Home Test 1 Solutions

Advanced Calculus II **Due February 20, 2008**

1. (Project 1, p. 219): You will show in this problem that if f is a continuous function on $[0,1]$, then $\| f \|_p \to \| f \|_\infty$ as $p \to \infty$.

(a) Show that for each p, $\| f \|_p \le \| f \|_\infty$.

> `Because` f `is continuous, we may use the theorem that says that` $f \le g$ `implies` $\int f \le \int g$. `Taking` g `to be the constant function` $g(x) = \sup_x |f(x)| = M$, `we have`
>
> $$\| f \|_p^p = \int_0^1 |f(x)|^p \, dx \le \int_0^1 M^p \, dx = M^p = (\sup_x |f(x)|)^p = \| f \|_\infty^p .$$
>
> `The result follows by taking` p`th roots.`

(b) Given that $f(x)$ is continuous on $[0,1]$, explain why $|f(x)|$ is also continuous on $[0,1]$. Then let x_0 be the point where $|f(x)|$ achieves its maximum. Explain why $|f(x_0)| = \| f \|_\infty$.

> `Suppose` $a \in [0,1]$ `has the property that` $f(x) > 0$. `Because` f `is continuous, there is some` δ `so that` $f(x) > 0$ `for all` x `within` δ `of` a. `In particular,` $f(x) = |f(x)|$ `on this` δ`-neighborhood. Thus` $|f(x)|$ `is continuous on this neighborhood. When` $f(a) < 0$, `we find that` $|f(x)| = -f(x)$ `on some open interval, so that again` $|f(x)|$ `is continuous there. The only issue is around those` a `for which` $f(a) = 0$. `For convenience, assume that` $f(x)$ `is negative just to the left of` a `and positive just to the right. The other cases are similar. Then because` f `is continuous, we have`
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> $$\lim_{x \to a^-} |f(x)| = -\lim_{x \to a^-} f(x) = 0 = \lim_{x \to a^+} f(x) = \lim_{x \to a^+} |f(x)|,$$
>
> `from which it follows that` f `is continuous at` a.
>
> `For the last statement, we have`
>
> $$|f(x_0)| = \sup_x |f(x)| = |\sup_x f(x)| = \| f \|_\infty .$$

(c) Assume that x_0 is not one of the endpoints of $[0,1]$, and let $\epsilon > 0$ be given. Explain why you can choose a $\mu > 0$ so that $|f(x)| \ge \| f \|_\infty - \epsilon$ for all $x \in [x_0 - \mu, x_0 + \mu]$.

`Because` $|f(x)|$ `is continuous at` x_0`, there is some` $\mu > 0$ `so that` $x \in [x_0 - \mu, x_0 + \mu]$ `implies that` $|f(x)| \in [f(x_0) - \epsilon, f(x_0) + \epsilon]$`. But because` $|f(x)|$ `is maximized at` x_0`, this is the same as saying that` $|f(x)| \in [f(x_0) - \epsilon, f(x_0)]$`. In other words,` $|f(x)| \geq \| f \|_\infty - \epsilon$ `for these` x `values.`

(d) Prove that $\int_0^1 |f(x)|^p \, dx \geq 2\mu(\| f \|_\infty - \epsilon)^p$ for all p.

`Using the last inequality (as well as the integral inequality theorem used in part (a)) we have`

$$\int_0^1 |f(x)|^p \, dx \geq \int_0^1 (\| f \|_\infty - \epsilon)^p \, dx = (\| f \|_\infty - \epsilon)^p \geq 2\mu(\| f \|_\infty - \epsilon)^p,$$

`where we've used the fact that` $\mu \leq \frac{1}{2}$ `in the last inequality (this is because we're looking at the interval` $[0,1]$`).`

(e) Show that for p large enough, $\| f \|_\infty \geq \| f \|_p \geq \| f \|_\infty - 2\epsilon$ and conclude that $\lim_{p\to\infty} \| f \|_p = \| f \|_\infty$.

`The first inequality is just part (a). For the second inequality we have from above that`

$$\| f \|_p = \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p} \geq (2\mu(\| f \|_\infty - \epsilon)^p)^{1/p} = (2\mu)^{1/p}(\| f \|_\infty - \epsilon).$$

`Because` $\mu < 1$`, we know that` $\lim_{p\to\infty} \mu^{1/p} = 1$`. Thus`

$$\lim_{p\to\infty} (2\mu)^{1/p}(\| f \|_\infty - \epsilon) = \| f \|_\infty - \epsilon.$$

`It follows that for` p `large enough we have`

$$(2\mu)^{1/p}(\| f \|_\infty - \epsilon) \geq \| f \|_\infty - 2\epsilon.$$

`The result follows.`

2. In this problem we will show that the 2–norm satisfies the triangle inequality. Recall that the 2–norm is defined by

$$\| f \|_2 = \left(\int_a^b [f(x)]^2 \, dx \right)^{1/2}.$$

(a) Show that for any real number λ, we have

$$0 \leq \| f + \lambda g \|_2^2 = \| f \|_2^2 + 2\lambda \int fg + \lambda^2 \| g \|_2^2 .$$

This is direct calculation. We have

$$\| f+\lambda g \|_2^2 = \int (f+\lambda g)^2 = \int (f^2+2\lambda fg+(\lambda g)^2) = \int f^2+2\lambda \int fg+\int \lambda^2 g^2$$

$$= \| f \|_2^2 + 2\lambda \int fg + \lambda^2 \| g \|_2^2 .$$

(b) Use part (a) to prove the Cauchy-Schwarz inequality: $\left(\int fg \right)^2 \leq \| f \|_2^2 \| g \|_2^2$. (Hint: Interpret the right-most expression in part (a) as a quadratic polynomial in λ. What does the fact that it is never negative tell you about its roots? What does this imply about the radical in the quadratic equation?)

We can write the quadratic polynomial in λ as $a\lambda^2 + b\lambda + c$, where $a = \| g \|_2^2$, $b = 2\int fg$, and $c = \| f \|_2^2$. Then the quadratic equation says that

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The fact that this quadratic is never negative means that it has either one or zero roots. This corresponds to the expression under the square root sign being either zero or negative. In other words, we have

$$b^2 - 4ac = 4\Big(\int fg\Big)^2 - 4 \| g \|^2 \| f \|^2 \leq 0 \quad \Rightarrow \quad \left(\int fg \right)^2 \leq \| f \|_2^2 \| g \|_2^2 .$$

(c) Prove the triangle inequality, $\| f+g \|_2 \leq \| f \|_2 + \| g \|_2$, by computing $\| f+g \|_2$ and applying the Cauchy-Schwarz inequality to the result.

$$\| f+g \|_2^2 = \int |f+g|^2 = \int f^2 + 2fg + g^2 = \int f^2 + 2\int fg + \int g^2$$

$$\leq \| f \|_2^2 + 2 \| f \|_2 \| g \|_2 + \| g \|^2 = (\| f \|_2 + \| g \|_2)^2.$$

The result follows by taking square roots.