Advanced Calculus II Test 2 Solutions

1. (a) Let ψ be a continuous function on $[a,b]$, and let ρ_ψ be the function on $C[a,b] \times C[a,b]$ defined by

$$\rho_\psi(f,g) = \int_a^b \psi(x)|f(x) - g(x)|\,dx.$$

What conditions on ψ are required for ρ_ψ to be a metric on $C[a,b]$? Justify your answer.

Symmetry is proven as follows:

$$\rho_\psi(f,g) = \int_a^b \psi(x)|f(x) - g(x)|\,dx = \int_a^b \psi(x)|g(x) - f(x)|\,dx = \rho_\psi(g,f).$$

The triangle inequality for ρ_ψ follows from that for absolute value, as well as the fact that $f \le g$ implies $\int f \le \int g$:

$$\rho_\psi(f,h) = \int_a^b \psi(x)|f(x)-h(x)|\,dx = \int_a^b \psi(x)|f(x)-g(x)+g(x)-h(x)|\,dx$$

$$\le \int_a^b \psi(x)(|f(x) - g(x)| + |g(x) - h(x)|)\,dx$$

$$= \int_a^b \psi(x)|f(x) - g(x)|\,dx + \int_a^b \psi(x)|g(x) - h(x)|\,dx = \rho_\psi(f,g) + \rho_\psi(g,h)$$

For positivity, we need that $\psi(x) > 0$ for all x. For then the same theorem about integrals as above implies that

$$\rho_\psi(f,g) = \int_a^b \psi(x)|f(x) - g(x)|\,dx \ge \int_a^b 0|f(x) - g(x)|\,dx = 0,$$

with equality if and only if the integrand is zero, i.e., if $f = g$ (this last statement requires that f and g are continuous).

(b) If X is a set, a *pseudometric* on X is a function $d : X \times X \to \mathbf{R}$ satisfying all the conditions of a metric, except that one is allowed to have $d(x,y) = 0$ for $x \ne y$. Suppose (X,d) is a pseudometric space. For all $x \in X$, let $\overline{x} = \{y \in X \mid d(x,y) = 0\}$, and let $\overline{X} = \{\overline{x} \mid x \in X\}$. Define $\overline{d} : \overline{X} \times \overline{X} \to \mathbf{R}$ by $\overline{d}(\overline{x}, \overline{y}) = d(x,y)$ for any $x \in \overline{x}$ and $y \in \overline{y}$. Show that this definition turns $(\overline{X}, \overline{d})$ into a metric space. (In particular, show that $\overline{d}$ is well-defined.)

We first show that $\overline{d}$ is well-defined. To that end, let x_1 and x_2 have $d(x_1, x_2) = 0$, and similarly choose y_1 and y_1. Then

$$\overline{d}(\overline{x_1}, \overline{y_1}) = d(x_1, y_1) \le d(x_1, x_2) + d(x_2, y_2) + d(y_2, y_1) = d(x_2, y_2) = \overline{d}(\overline{x_2}, \overline{y_2}).$$

A similar argument shows that $\overline{d}(\overline{x_2}, \overline{y_2}) \le \overline{d}(\overline{x_1}, \overline{y_1})$, so that in fact these quantities are equal. This shows that $\overline{d}$ is well-defined. Now note that

$$\overline{d}(\overline{x}, \overline{y}) = d(x, y) \ge 0,$$

and

$$\overline{d}(\overline{x}, \overline{y}) = 0 \Leftrightarrow d(x, y) = 0 \Leftrightarrow \overline{x} = \overline{y}.$$

This proves strict positivity. Symmetry follows immediately from symmetry in d, as does the triangle inequality.

2. Let $\mathcal{M}$ be the set of continuous functions on $\mathbf{R}$ that vanish outside a finite interval. In other words, for each $f \in \mathcal{M}$, there is some interval $[a, b]$ so that $f(x) = 0$ for all $x \notin [a, b]$. (Note that the interval depends on f, so different functions may have different intervals.)

(a) Show that $\mathcal{M}$ is not complete in the sup norm (find a Cauchy sequence that doesn't converge in $\mathcal{M}$).

Here is an example of such a sequence. Define

$$f_n(x) = \begin{cases} e^{-x^2} & x \in [-n, n] \\ 0 & x \notin [-n-1, n+1]. \end{cases}$$

Note that these functions are not defined on $[-n-1, -n)$ or $(n, n+1]$. We define $f_n(x)$ on these intervals to be the straight line joining $f(-n)$ to $f(-n-1)$ and $f(n)$ to $f(n+1)$, respectively. This makes each $f_n(x)$ continuous. In particular, $f_n(x) \in \mathcal{M}$ for all n. It is straightforward to show that $f_n(x) \to e^{-x^2}$ in the sup norm, but $e^{-x^2} \notin \mathcal{M}$.

(b) Let $C_0(\mathbf{R})$ be the set of continuous functions on $\mathbf{R}$ that go to zero at $\pm\infty$. Show that $C_0(\mathbf{R})$ is complete in the sup norm.

Suppose f_n is a Cauchy sequence of functions in $C_0(\mathbf{R})$. Because the set of continuous functions on $\mathbf{R}$ is complete in the sup norm, we know that there is some continuous limit f for this sequence. We have only to show that $\lim_{x\to\pm\infty} f(x) = 0$. Let $\epsilon > 0$ be given. The fact that the f_n converge to f in the sup norm implies that there is some N so that $\| f - f_n \|_\infty < \epsilon$ whenever $n \ge N$. By definition of the sup norm, this means that for $n \ge N$, we have that $|f(x) - f_n(x)| < \epsilon$ for all x. In particular, $\lim_{x\to\pm\infty} |f(x) - f_n(x)| \le \epsilon$. But then $\lim_{x\to\pm\infty} ||f(x)| - |f_n(x)|| \le \lim_{x\to\pm\infty} |f(x) - f_n(x)| < \epsilon$. It follows that $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} f_n(x) = 0$.

(c) Prove that $\mathcal{M}$ is dense in $C_0(\mathbf{R})$.

Suppose f is a function in $C_0(\mathbf{R})$. We need to find a sequence of functions f_n in $\mathcal{M}$ that converge to f in the sup norm. For this, we mimic the example in part (a). Define

$$f_n(x) = \begin{cases} f(x) & x \in [-n, n] \\ 0 & x \notin [-n-1, n+1], \end{cases}$$

and extend linearly across the missing intervals as before. Then the sup norm of the difference between f_n and f is bounded above by $L_n = 2\sup\{|f(x)| \mid x \in [n, n+1] \cup [-n-1, -n]\}$. But $L_n \to 0$ as $n \to \infty$. The result follows.

3. The Riemann zeta function is the following: $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$. This function is continuous for $x > 1$. Let $\mathbb{P}$ denote the set of all prime numbers. Prove that for all $x > 1$ we have

$$\zeta(x) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^x}\right)^{-1}.$$

For each prime p, we have the following geometric series:

$$\frac{1}{1 - \frac{1}{p^x}} = 1 + \frac{1}{p^x} + \frac{1}{p^{2x}} + \frac{1}{p^{3x}} + \cdots.$$

Thus the infinite product above becomes

$$\prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^x}\right)^{-1} = \prod_{p \in \mathbb{P}} \left[1 + \frac{1}{p^x} + \frac{1}{p^{2x}} + \frac{1}{p^{3x}} + \cdots\right]$$

$$= \left(1 + \frac{1}{2^x} + \frac{1}{2^{2x}} + \frac{1}{2^{3x}} + \cdots\right)\left(1 + \frac{1}{3^x} + \frac{1}{3^{2x}} + \frac{1}{3^{3x}} + \cdots\right)\left(1 + \frac{1}{5^x} + \frac{1}{5^{2x}} + \frac{1}{5^{3x}} + \cdots\right)\cdots.$$

Fix a number n and let $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ be its prime decomposition. The expansion of the product above consists of summands which are products whose factors are chosen one from each infinite series. Thus the term $\frac{1}{n^x}$ shows up as a summand in the expansion when the term from the series corresponding to p_j is the m_j-th summand, and all other terms are 1. It follows that

$$\zeta(x) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^x}\right)^{-1}.$$