Series Summary

A *power series* centered at c is a series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots.$$

One key example of a power series is the *Taylor series* centered at c of an infinitely differentiable function $f(x)$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots.$$

For any power series, there is some number R (with $0 \le R \le \infty$) so that plugging in any x-values with $c - R < x < c + R$ yields an absolutely convergent series, while plugging in any x-values with $x < c - R$ or $x > c + R$ yields a divergent series. This R is called the *radius of convergence*, and is most often found using the ratio test (or sometimes the root test):

Ratio Test: For $\sum b_n$, we have that

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} \begin{cases} < 1 & \Rightarrow \quad \text{absolute convergence} \\ > 1 & \Rightarrow \quad \text{divergence} \\ = 1 & \Rightarrow \quad \text{no information} \end{cases}$$

When $\sum b_n$ is a power series, each b_n has an x in it, so that the inequality in the ratio test turns into an inequality in x, allowing you to find the radius of convergence. The **root test** works the same way except that the quantity whose limit you compute is $\sqrt[n]{|b_n|}$.

Neither the ratio nor the root test tells you anything about what happens when $x = c \pm R$. To test for convergence for these values of x, you must explicitly plug in each value and analyze the resulting series using one of the following tests:

Integral Test: If $f(x)$ is (eventually) continuous, positive, and decreasing, and $f(n) = b_n$ for each n, then convergence or divergence of $\sum b_n$ is determined by convergence or divergence of

$$\int_a^{\infty} f(x)\, dx.$$

Comparison Test: Suppose $0 \le a_n \le b_n$ for all n. Then if $\sum a_n$ diverges, so does $\sum b_n$, while if $\sum b_n$ converges, so does $\sum a_n$. (For positive numbers, bigger than infinite is infinite, and smaller than finite is finite.)

Limit Comparison Test: Given two positive series $\sum a_n$ and $\sum b_n$, if

$$0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty,$$

then the two series converge or diverge together.

Note that all three of the above tests work only for POSITIVE series.

Alternating Series Test: Given an alternating series

$$\sum(-1)^n b_n = b_0 - b_1 + b_2 - b_3 + b_4 - \cdots ,$$

with $b_n \geq 0$, the series converges as long as

$$b_0 \geq b_1 \geq b_2 \geq b_3 \geq \cdots \quad \text{and} \quad \lim_{n\to\infty} b_n = 0.$$

Geometric Series: Given a geometric series

$$\sum_{n=0}^{\infty} ar^n$$

if $|r| < 1$, then the series converges to $\dfrac{a}{1-r}$. Otherwise it diverges.