Test 1 Solutions   Calculus II   February 13, 2009

1. $\int_{-1}^{\infty} xe^{-2x}\,dx$

We use integration by parts, with $u = x$ and $dv = e^{-2x}\,dx$. Thus $du = dx$, and $v = -\frac{1}{2}e^{-2x}$ (this may be found using the substitution $w = -2x,\ dw = -2\,dx$). Thus the integration by parts formula tells us that

$$\int xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\,dx.$$

We already evaluated this new integral in finding v above, so we have that

$$\int xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}.$$

Evaluating this expression from $x = -1$ to $x = n$ gives

$$\int_{-1}^{n} xe^{-2x}\,dx = \left(-\frac{1}{2}ne^{-2n} - \frac{1}{4}e^{-2n}\right) - \left(\frac{1}{2}e^{2} - \frac{1}{4}e^{2}\right)$$
$$= -\frac{2n+1}{4e^{2n}} - \frac{e^2}{4}.$$

Taking the limit of this expression as $n \to \infty$, the first term goes to zero, so we finally have

$$\int_{-1}^{\infty} xe^{-2x}\,dx = -\frac{e^2}{4}.$$

2. $\int \cos^3 x \sin^8 x\,dx$

Because there are an odd number of cosines, we turn all but one cosine into sines (two at a time), with the goal of substituting $u = \sin x$, with the remaining cosine as our du. Thus we have

$$\int \cos^3 x \sin^8 x\,dx = \int (1-\sin^2 x)\sin^8 x \cos x\,dx = \int (1-u^2)u^8\,du$$
$$\int (u^8 - u^{10})\,du = \frac{u^9}{9} - \frac{u^{11}}{11} + C = \frac{1}{9}\sin^9 x - \frac{1}{11}\sin^{11} x + C.$$

3. $\int_1^4 \frac{1}{x+x^{-1}}\,dx$

We first multiply the numerator and denominator by x, to simplify the expression, obtaining

$$\int_1^4 \frac{1}{x+x^{-1}}\,dx = \int_1^4 \frac{x}{x^2+1}\,dx.$$

For this integral, we apply u-substitution, with $u = x^2+1$ and $du = 2x\,dx$. (Trig substitution will also work, but it's more complicated.) Thus we have

$$\int \frac{1}{x+x^{-1}}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C.$$

Evaluating from $x = 1$ to $x = 4$, we have

$$\int_1^4 \frac{1}{x+x^{-1}}\,dx = \frac{1}{2}(\ln(17) - \ln(2)).$$

4. $\displaystyle\int \frac{4x+4}{(x-1)(x+3)^2}\,dx$

We use the partial fraction technique, for which the decomposition is

$$\frac{4x+4}{(x-1)(x+3)^2} = \frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{(x+3)^2}.$$

Clearing denominators, we have

$$4x + 4 = a(x+3)^2 + B(x-1)(x+3) + C(x-1).$$

Plugging in roots $x = 1$ and $x = -3$ shows that $A = \frac{1}{2}$ and $C = 2$. To find B we plug in any other value for x and use these values of A and C, finding that $B = -\frac{1}{2}$. We now have

$$\int \frac{4x+4}{(x-1)(x+3)^2}\,dx = \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{1}{x+3}\,dx + 2\int \frac{1}{(x+3)^2}\,dx$$
$$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+3| - \frac{2}{x+3},$$

where these integrals are evaluated using the substitutions $u = x-1$ and $u = x+3$ as appropriate.

5. $\displaystyle\int \frac{1}{x(x^2-1)^{3/2}}\,dx$

This is a trig substitution problem, where we use $x = \sec\theta$, so $dx = \sec\theta\tan\theta$. This gives us

$$\int \frac{1}{x(x^2-1)^{3/2}}\,dx = \int \frac{\sec\theta\tan\theta}{\sec\theta(\sec^2\theta-1)^{3/2}}\,d\theta = \int \frac{\tan\theta}{(\tan^2\theta)^{3/2}}\,d\theta$$
$$= \int \frac{\tan\theta}{\tan^3\theta}\,d\theta = \int \frac{1}{\tan^2\theta}\,d\theta = \int \cot^2\theta\,d\theta = \int (\csc^2\theta - 1)\,d\theta$$
$$= \cot\theta - \theta + C = \frac{1}{\sqrt{x^2-1}} - \sec^{-1}x + C,$$

where this last expression is obtained using a right triangle with hypotenuse x, adjacent leg 1, and opposite leg $\sqrt{x^2-1}$.