Test 2 Calculus II March 13, 2009

1. Find the Taylor series centered at zero for the indefinite integral $\int \frac{1}{1-2x^3}\,dx$.

We begin with a known series, then substitute, then integrate:

$$\frac{1}{1-x} \quad : \quad \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$\frac{1}{1-2x^3} \quad : \quad \sum_{n=0}^{\infty} (2x^3)^n = \sum_{n=0}^{\infty} 2^n x^{3n} = 1 + 2x^3 + 4x^6 + 8x^9 + 16x^{12} + \cdots$$

$$\int \frac{1}{1-2x^3}\,dx \quad : \quad \sum_{n=0}^{\infty} \frac{2^n x^{3n+1}}{3n+1} = x + \frac{2x^4}{4} + \frac{4x^7}{7} + \frac{8x^{10}}{10} + \frac{16x^{13}}{13} + \cdots .$$

2. Find all values of x for which the following power series converges:

$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{2^n\sqrt{n+3}} = \frac{1}{\sqrt{3}} + \frac{x-3}{4\sqrt{5}} + \frac{(x-3)^2}{8\sqrt{6}} + \frac{(x-3)^3}{16\sqrt{7}} + \cdots$$

We use the ratio test:

$$\left| \frac{\frac{(x-3)^{n+1}}{2^{n+1}\sqrt{(n+1)+3}}}{\frac{(x-3)^n}{2^n\sqrt{n+3}}} \right| = \left| \frac{(x-3)^{n+1} 2^n \sqrt{n+3}}{(x-3)^n 2^{n+1}\sqrt{n+4}} \right| = \left| \frac{x-3}{2}\sqrt{\frac{n+3}{n+4}} \right| \xrightarrow{n\to\infty} \left| \frac{x-3}{2} \right| .$$

Setting this less than one yields

$$|x-3| < 2 \qquad \Leftrightarrow \qquad 1 < x < 5.$$

Now we check the endpoints. When $x = 1$, we have

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{2^n\sqrt{n+3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n\sqrt{n+3}}.$$

This series converges by the alternating series test, because

$$\frac{1}{2^n\sqrt{n+3}} > \frac{1}{2^{n+1}\sqrt{n+4}} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{2^n\sqrt{n+3}} = 0.$$

When $x = 5$, we have

$$\sum_{n=0}^{\infty} \frac{2^n}{2^n\sqrt{n+3}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+3}},$$

which diverges by limit comparison with $\sum \frac{1}{\sqrt{n}}$. Thus the values of x for which the series converges are $1 \le x < 5$.

3. Find the Taylor series for $f(x) = x^3 - 2x + 4$ centered at $c = 2$.

We compute

$$\begin{aligned} f(2) &= 8-4+4=8 \\ f'(2) &= 12-2=10 \\ f''(2) &= 12 \\ f'''(2) &= 6 \end{aligned}$$

All higher derivatives are zero. Plugging into the Taylor series formula, we compute

$$T(x)=8+10(x-2)+\frac{12}{2}(x-2)^2+\frac{6}{3!}(x-2)^3=8+10(x-2)+6(x-2)^2+(x-2)^3.$$

4. What fraction equals $0.181818\ldots$?

We write this as the geometric series

$$\frac{18}{10^2}+\frac{18}{10^4}+\frac{18}{10^6}+\frac{18}{10^8}+\cdots$$

which has $a=\frac{18}{100}$ and $r=\frac{1}{100}$. Thus the series converges to

$$\frac{\frac{18}{100}}{1-\frac{1}{100}}=\frac{18}{99}=\frac{2}{11}.$$

5. (a) What is the formula for the coefficient of x^{15} in the Taylor series for a function $f(x)$ centered at zero?

The Taylor series centered at zero has general summand $\dfrac{f^{(n)}(0)}{n!}x^n$, so the coefficient of x^{15} is $\dfrac{f^{(15)}(0)}{15!}$.

(b) Use substitution into a known series to determine explicitly the coefficient of x^{15} in the Taylor series for $\sin(x^3)$ centered at zero.

$$\begin{aligned} \sin(x) &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} &= x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots \\ \sin(x^3) &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+3}}{(2n+1)!} &= x^3-\frac{x^9}{3!}+\frac{x^{15}}{5!}-\frac{x^{21}}{7!}+\cdots, \end{aligned}$$

so the coefficient of x^{15} is $\dfrac{1}{5!}$.

(c) Compute the value of the fifteenth derivative of $\sin(x^3)$ at zero by setting the two answers above equal to one another and solving.

The general formula says that the coefficient is $\dfrac{f^{(15)}(0)}{15!}$, while we've explicitly computed the coefficient to be $\dfrac{1}{5!}$. Thus

$$\frac{f^{(15)}(0)}{15!}=\frac{1}{5!}\quad\Leftrightarrow\quad f^{(15)}(0)=\frac{15!}{5!}=10,897,286,400.$$