Test 3

Name: **Calculus II**

ID: **April 17, 2009**

Instructions: Please be sure to write neatly, show your work, and put your answer in a box.

1. Consider a pool of the shape indicated below filled with water. Express the work done in pumping the water out of the pool through the spout as an integral in a single variable. Do not evaluate the integral. (Distance units are in meters, gravity is 9.8 m/s^2 and the density of water is 1000 kg/m^3.)

Work is force times distance. We need to integrate because different molecules of water have different distances to travel. Thus we slice the water into pieces that are thin enough in some direction or another so that each molecule in such a slice has roughly the same distance to travel. In other words, we slice horizontally. This gives us rectangular slices.

Let's assume the x-axis runs vertically with the origin at the bottom of the pool, and consider a slice at height x. Every such slice has length 4, but the widths vary with x. Using similar triangles we discover that

$$\frac{6}{2} = \frac{w}{2-x} \qquad \Rightarrow \qquad w = 3(2-x) = 6 - 3x.$$

Thus the area of a slice is $\ell w = 4(6-3x) = 24 - 12x$. Thus the volume of a thin slab is $(24-12x)\Delta x$. The distance such a slab has to travel to get to the top of the spout is $3 - x$. Thus the work done in lifting such a slab is

$$F \cdot d = m \cdot g \cdot d = \rho \cdot V \cdot g \cdot d = (1000)(24-12x)\Delta x(9.8)(3-x).$$

Adding up the work for each such slab and taking limits as the slabs get thin leads to the integral

$$W = 9800 \int_0^2 (24-12x)(3-x)\,dx.$$

2. Let $\mathcal{R}$ denote the region bounded by the graph of $y = x^2$ and the line $y = 4$. Express each of the following quantities as an integral in a single variable. Do not evaluate the integrals.

(a) The area of $\mathcal{R}$.

We integrate the top curve minus the bottom curve. This gives the area under the line minus the area under the parabola, leaving the area between Thus we have

$$A = \int_{-2}^{2} (4 - x^2)\, dx.$$

(b) The volume of the solid obtained by rotating $\mathcal{R}$ about the x-axis.

Rotating a thin vertical slice of the region gives a washer, the volume of which is

$$(\pi R^2 - \pi r^2)\Delta x = (\pi 4^2 - \pi (x^2)^2)\Delta x.$$

Adding up such volumes and taking limits as the slices get thin yields

$$V = \pi \int_{-2}^{2} (16 - x^4)\, dx.$$

(c) The surface area of the solid in part (b). (Hint: Use the surface area formula for the area coming from the rotated parabola; rotating the line just gives you a cylinder.)

Rotating a small piece of the parabola gives a cylinder-like object whose area is approximately

$$2\pi r L = 2\pi x^2 \sqrt{1 + (2x)^2}\Delta x.$$

Thus the surface area coming from the parabola is

$$SA = 2\pi \int_{-2}^{2} x^2 \sqrt{1 + 4x^2}\, dx.$$

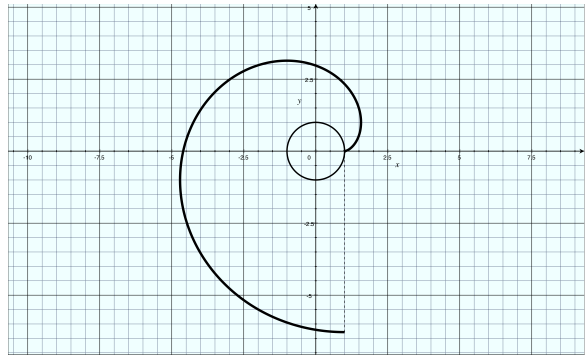
Applying a similar technique to the line, or just using the formula for area of a cylinder, gives the area coming from the line as

$$SA = \int_{-2}^{2} 8\pi\, dx.$$

The total surface area is the sum of these two quantities.

3. If you unwind thread from a stationary circular spool, keeping the thread taut at all times, the the endpoint traces a curve called the *involute* of the circle. For a circle of radius one centered at the origin, the involute can be parametrized by $c(t) = (\cos t + t \sin t, \sin t - t \cos t)$. Find the length of the involute for $0 \leq t \leq 2\pi$.

(Hint: The final integrand is simple; do your algebra, trig, and differentiation carefully.)



We compute

$$\frac{dx}{dt} = -\sin t + \sin t + t\cos t = t\cos t \qquad \frac{dy}{dt} = \cos t - \cos t + t\sin t = t\sin t.$$

Squaring each of these and adding gives

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = t^2(\cos^2 t + \sin^2 t) = t^2.$$

It follows that the arclength is

$$L = \int_0^{2\pi} \sqrt{t^2}\, dt = \int_0^{2\pi} t\, dt = 2\pi^2.$$

4. Calculate the population within a 5-mile radius of the city center is the radial population density is $\rho(r) = 4(2 + r^2)^{1/3}$ (in thousands per square mile).

The density is a function of radial distance, so we chop the region into pieces within which the radial distance is roughly constant. These are thin washers, the area of which is approximately $2\pi r \Delta r$. Multiplying this area by the given density gives the approximate number of people in that washer. Adding and taking limits gives

$$P = \int 2\pi r 4(2 + r^2)^{1/3}\, dr = 4\pi \int u^{1/3}\, du = 4\pi \frac{u^{4/3}}{4/3} = 3\pi(2 + r^2)^{4/3}.$$

Plugging in the limits $r = 5$ and $r = 0$, we obtain

$$P = 3\pi(27)^{4/3} = 243\pi.$$