1. Use the Brouwer fixed point theorem to prove the following: Every 3×3 matrix with positive real entries has an eigenvector with positive eigenvalue.

> Hint: Let $\Delta = \{(x, y, z) \mid x + y + z = 1,\ x, y, z \geq 0\} \subset \mathbf{R}^3$. Then Δ is topologically a disk. What happens when the matrix acts on the vectors in Δ via matrix multiplication? How can you translate that into a map $\Delta \to \Delta$? What does Brouwer then tell you?

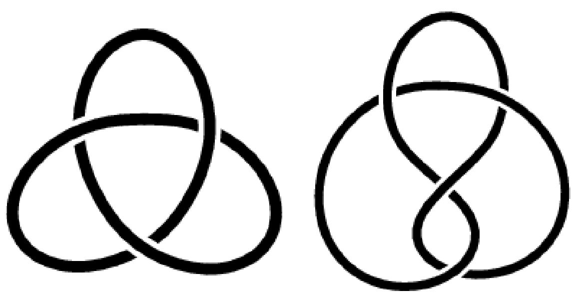
2. (Hatcher #22, p. 55). Suppose $K : S^1 \to \mathbf{R}^3$ is a smooth knot in $\mathbf{R}^3$.

(a) Use van Kampen's theorem and the hints in Hatcher to show that $\pi_1(\mathbf{R}^3 - K)$ has a presentation of the form $\langle x_1, \ldots, x_n \mid x_i x_j x_i^{-1} = x_k \rangle$ (the relation assumes $j \neq k$).

> Hint: To see better that $\mathbf{R}^3 - K$ deformation retracts onto X, glue a halfspace $\mathbf{R}^2 \times [0, \infty)$ onto the bottom and top of X (note that the top of X is also a topological plane once the squares are attached). This new space is homeomorphic to $\mathbf{R}^3$ minus an open neighborhood of K.

(b) Use this presentation to show that the abelianization of $\pi_1(\mathbf{R}^3 - K)$ is $\mathbf{Z}$ regardless of K.

(c) Write an explicit presentation for $\pi_1(\mathbf{R}^3 - K)$ for the two knots shown below:



3. Consider the map $p : \mathbf{R} \times \mathbf{R}_+ \to \mathbf{R}^2 - \{\mathbf{0}\}$ given by

$$p(t, r) = re^{2\pi i t}.$$

(a) Show that p is a covering map.

(b) For each of the following parametrized paths, sketch the path in $\mathbf{R}^2 - \{\mathbf{0}\}$ as well as its lift to $\mathbf{R} \times \mathbf{R}_+$:

$$f(t) = \langle 2 - t, 0 \rangle, \qquad g(t) = \langle (1 + t) \cos 2\pi t, (1 + t) \sin 2\pi t \rangle.$$

4. We know that every homeomorphism $f : X \to X$ induces an isomorphism $f_* : \pi_1(X) \to \pi_1(X)$. It turns out that the converse is true in some special cases, in particular when X is a closed surface (but *not* when it is a surface with boundary). We will prove this in the case of the torus.

Let $T = S^1 \times S^1$ be the torus. We will show that every isomorphism $g : \pi_1(T) \to \pi_1(T)$ has the property that $g = f_*$ for some homeomorphism $f : T \to T$.

(a) Show that any group homomorphism $g : \mathbf{Z}^2 \to \mathbf{Z}^2$ extends to a linear map $L_g : \mathbf{R}^2 \to \mathbf{R}^2$.

> Hint: Once you know what $g(1,0)$ and $g(0,1)$ are, you know all of g (by the homomorphism assumption), and then you know all of L_g (by the linearity assumption).

(b) Let A_g denote the 2×2 matrix corresponding to L_g (with respect to the standard basis). Show that g is an isomorphism if and only if $\det(A_g) = \pm 1$.

(c) Let $GL(2,\mathbf{Z})$ denote the group of 2×2 integer matrices with determinant ± 1. Show that any matrix in $GL(2,\mathbf{Z})$ can be written as a finite product of the following five matrices in $GL(2,\mathbf{Z})$:

$$\begin{bmatrix} 1 & \pm 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ \pm 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(d) Let $\mathrm{Aut}(T)$ denote the group of self-homeomorphisms $T \to T$. Show that the map $\mathrm{Aut}(T) \to GL(2,\mathbf{Z})$ given by $f \mapsto f_*$ is surjective by finding preimages for each of the five matrices above.

5. (In this problem, we will use the association of $\mathrm{Aut}(T)$ with $GL(2,\mathbf{Z})$ discovered in the previous problem.)

For integers λ and μ let $C_{\lambda,\mu}$ denote the loop on the torus given by $C_{\lambda,\mu} : [0,1] \to S^1 \times S^1$, $C_{\lambda,\mu}(t) = (e^{2\lambda\pi i t}, e^{2\mu\pi i t})$. Geometrically $C_{\lambda,\mu}$ winds λ times around the long way (the longitude), and μ times around the short way (the meridian).

(a) Let $p : \mathbf{R}^2 \to T$ be the usual covering projection. Show that the lift of $C_{\lambda,\mu}$ to $\mathbf{R}^2$ starting at the origin ends at (λ, μ).

(b) Assume $\gcd(\lambda,\mu) = 1$. Find a linear map $A \in GL(2,\mathbf{Z})$ sending (λ,μ) to $(1,0)$.

(c) Use your answer to part (b) to show that the number of times the curve $C_{a,b}$ intersects the curve $C_{c,d}$ is $ad - bc$.

> Hint: The number of times that $C_{a,b}$ intersects $C_{c,d}$ is equal to the number of lifts of $C_{a,b}$ intersecting a single lift $\widetilde{C}_{c,d}$ of $C_{c,d}$ (because intersections lift to intersections). Acting on the whole picture by A preserves the number of intersections, so we're left with finding the number of intersections of $A \cdot \widetilde{C}_{c,d}$ with the lines in $A(p^{-1}(C_{a,b}))$. If A sends (a,b) to $(1,0)$, what are the lines in $A(p^{-1}(C_{a,b}))$? How can you count the number of these lines hit by $A \cdot \widetilde{C}_{c,d}$?

(c) Suppose you have a clock with two identical hands whose positions can be measured with absolute accuracy. How many different times are there where you can know via such a measurement what time it is?

> Hint: Each hand traces out a circle. Thus the position (hour, minute) can be thought of as a point on the torus $S^1 \times S^1$. In the time from midnight to noon, what curve $C(a,b)$ is traced out on the torus by the hand positions (hour, minute)? What about (minute, hour)? What does it mean when these two curves intersect?

6. Let $SO(3)$ denote the group of rotations of $\mathbf{R}^3$ about the origin. Algebraically, $SO(3)$ is the group of 3×3 orthogonal matrices with determinant one: $A \in SO(3)$ means that A can be written as consisting of three column vectors $A = [\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3]$, which are pairwise orthogonal unit vectors in $\mathbf{R}^3$. In particular, $\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}$, the Dirac delta function. Also, the orientation of the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is consistent with that of $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. (In fact, the vectors $\mathbf{v}_i$ are the images of the standard basis vectors under the rotation corresponding to A.)

We can give the space of all 3×3 matrices a topology by identifying it with $\mathbf{R}^9$ coordinatewise. Thus $SO(3)$ inherits a topology where an open neighborhood of a matrix in $SO(3)$ consists of matrices A in $SO(3)$ whose entries are contained in open neighborhoods in $\mathbf{R}$ of the entries of A.

(a) Show that the universal cover of $SO(3)$ is S^3 and that $\pi_1(SO(3))$ is $\mathbf{Z}_2$.

> Hint: An element of $SO(3)$, thought of as a rotation, is determined by the angle and axis of rotation. Think of the angle parameter as $\theta \in [0, 2\pi]$, and the axis as a unit vector (i.e., in S^2). Then $SO(3)$ is a topological quotient space of $[0, 2\pi] \times S^2$.

(b) Show that there is no non-vanishing tangent vector field on S^2 as follows:

Suppose there is such a vector field. Since all the vectors are nonzero, we may divide by their lengths to obtain a *unit* tangent vector field on S^2. Thus we have a map

$$\mathbf{r} : S^2 \to S^2$$

with $\mathbf{r}(\mathbf{v}) \cdot \mathbf{v} = 0$ for all $\mathbf{v} \in S^2$ (note that $\mathbf{v} \in S^2$ is equivalent to $\| \mathbf{v} \| = 1$, and the dot product condition ensures that $\mathbf{r}(\mathbf{v})$ is tangent to the sphere at $\mathbf{v}$ by making it perpendicular to the radial vector $\mathbf{v}$).

Let $\mathbf{n}(\mathbf{v}) = \mathbf{v} \times \mathbf{r}(\mathbf{v})$, and define a 3×3 matrix by

$$A = [\mathbf{v},\ \mathbf{r}(\mathbf{v}),\ \mathbf{n}(\mathbf{v})] \in SO(3).$$

Define $\phi : S^1 \times S^2 \to SO(3)$ by $\phi(\theta, \mathbf{v}) = R_\theta \circ A(\mathbf{v})$, where R_θ is rotation about the line determined by $\mathbf{v}$ through angle θ (measured from $\mathbf{r}(\mathbf{v})$ to $\mathbf{n}(\mathbf{v})$ and with $-\pi \leq \theta \leq \pi$).

Show that ϕ is continuous, one-to-one, and onto, hence a homeomorphism. Why is this a contradiction?