Basic Definitions

A **series** is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots .$$

This series **converges** if the sequence of **partial sums**

$$s_n = a_1 + a_2 + \cdots + a_n$$

has a finite limit. In this case we say that the **sum** of the series is equal to this limit, and write

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n \quad \text{if this limit exists and is finite.}$$

If the series does not converge, we say it **diverges**. The series $\sum a_n$ **converges absolutely** if it still converges after all negative signs are made positive. In other words,

$$\sum_{n=1}^{\infty} a_n \quad \text{“converges absolutely” means that} \quad \sum_{n=1}^{\infty} |a_n| \quad \text{converges.}$$

Any series that converges absolutely also converges in the ordinary sense. On the other hand, there are series that converge, yet do not converge absolutely. These are called **conditionally convergent** series.

Convergence Tests

1. Apply the nth term test.

   If $\lim_{n\to\infty} a_n \neq 0$, *then the series diverges. If* $\lim_{n\to\infty} a_n = 0$, *the series may or may not converge. Proceed to Step 2.*

2. Check if the series is a geometric series or a p-series.

   If it's a geometric series $\sum ar^n$ *and* $|r| < 1$, *then it converges to* $\dfrac{a}{1-r}$.

   If it's a p*-series* $\sum \dfrac{1}{n^p}$ *and* $p > 1$, *then it converges. Otherwise, it diverges.*

3. If $\sum a_n$ is a positive term series, use one of the following tests.

   - Basic comparison test (usually compare to a p-series or a geometric series).

     If there is a convergent series that is bigger than your series, then your series also converges. If there is a divergent series that is smaller than your series, then your series also diverges. “Bigger than infinite is still infinite, smaller than finite is still finite.”

- Limit comparison test (as above).

  If the limit $\lim(a_n/b_n)$ *of the ratio of the corresponding terms of two series exists and is finite, then the series either both converge or both diverge.*

- Ratio test (useful when there is an $n!$ or a c^n).

  Consider the limit $\lim |a_{n+1}/a_n|$ *of the ratio of consecutive terms of your series. If this limit is less than* 1*, the series converges (absolutely). If this limit is greater than* 1*, the series diverges. If this limit equals* 1*, you know nothing.*

- Root test (useful when there is an n^n or something similar).

  Consider the limit $\lim \sqrt[n]{|a_n|}$ *of the nth roots of the nth terms of your series. If this limit is less than* 1*, the series converges (absolutely). If this limit is greater than* 1*, the series diverges. If this limit equals* 1*, you know nothing.*

- Integral test (must have terms a_n that are decreasing toward 0 and can be integrated).

  If $a_n = f(n)$ *for some function defined for* $x \geq 1$ *that is positive, continuous, and decreasing, then your series and* $\int_1^\infty f(x)\,dx$ *converge or diverge together.*

4. If $\sum a_n$ is an alternating series, use one of the following tests.

   - Alternating series test.

     If the absolute values $|a_n|$ *of the terms in an alternating series are decreasing and limit to* 0*, then the series converges.*

   - A test from Step 3 applied to $\sum |a_n|$.

5. For a power series $\sum c_n x^n$, use the ratio or root test to find the interval of convergence; then use the above tests to check for convergence at the endpoints of the interval.

This is adapted from §4.11 "Which test to apply when?" from *How to Ace the Rest of Calculus,* by Adams, Hass, and Thompson. Please buy their book.