

1. Use known series to find Taylor series (centered at $x = 0$) for the following functions. Find at least the first five non-zero terms of the series. You do not have to write the series in summation notation. Also state the radius of convergence of the series.

(a) $\int \frac{1}{1+x^5} dx$

We begin with the series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots,$$

which converges when $|x| < 1$. Substituting in $-x^5$, we have that

$$\frac{1}{1+x^5} = 1 - x^5 + x^{10} - x^{15} + x^{20} - \dots,$$

which converges for $|-x^5| < 1$, or $-1 < x < 1$. Integrating term by term, we find that for $-1 < x < 1$, we have

$$\int \frac{1}{1+x^5} dx = x - \frac{x^6}{6} + \frac{x^{11}}{11} - \frac{x^{16}}{16} + \frac{x^{21}}{21} - \dots$$

(b) $\cosh(x) = \frac{e^x + e^{-x}}{2}$

We begin with the series

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

and find that

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots$$

Adding these together, note that the even powered terms double, while the odd powered terms cancel, so we have

$$e^x + e^{-x} = 2\left(1 + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \dots\right),$$

so that finally we have

$$\cosh(x) = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \dots$$

Because every series involved converges for all x , this series also converges for all x . Note that this is the same as the ordinary cosine series, except with no negative signs, at least partially justifying giving this function the similar name of hyperbolic cosine.

2. (a) Find the degree two Taylor approximation for \sqrt{x} centered at $x = 4$ and use this approximation to estimate $\sqrt{5}$.

The degree two Taylor approximation at $x = 4$ is of the form

$$f(x) \approx f(4) + f'(4)(x - 4) + \frac{f''(4)}{2}(x - 4)^2.$$

Letting $f(x) = \sqrt{x} = x^{1/2}$, we have that

$$f'(x) = \frac{1}{2}x^{-1/2} \quad \text{and} \quad f''(x) = \frac{-1}{4}x^{-3/2}.$$

Plugging in $x = 4$ to these we find that

$$f(4) = 2, \quad f'(4) = \frac{1}{4}, \quad \text{and} \quad f''(4) = \frac{-1}{32}.$$

Thus we have that

$$f(x) \approx 2 + \frac{1}{4}(x - 4) - \frac{1}{64}(x - 4)^2,$$

from which we obtain

$$\boxed{f(5) = \sqrt{5} \approx 2 + \frac{1}{4} - \frac{1}{64} = \frac{143}{64}.$$

(b) Bound the error (remainder) of this approximation (you will need to use the fact that $x^{-5/2}$ is a decreasing function, so $x^{-5/2}$ is biggest when x is smallest).

The remainder is given by

$$R_3(5) = \frac{f'''(c)}{3!}(5 - 4)^3 = \frac{f'''(c)}{6}$$

for some c between 4 and 5. Note that $f'''(x) = \frac{3}{8}x^{-5/2}$, so $f'''(c) = \frac{3}{8}c^{-5/2}$. As mentioned in the hint, this is largest when c is smallest. Because c has to be between 4 and 5, this means that this is largest when $c = 4$. Thus we have

$$\boxed{R_3(5) = \frac{f'''(c)}{6} \leq \frac{4^{-5/2}}{6} = \frac{1}{512}.$$

3. Find the sum of the following series: $6 + 4 + \frac{8}{3} + \frac{16}{9} + \frac{32}{27} + \dots$.

The only type of series we can always sum is a geometric series, so we look for a common ratio between terms. To this end note that

$$\frac{4}{6} = \frac{8/3}{4} = \frac{16/9}{8/3} = \frac{32/27}{16/9} = \frac{2}{3}.$$

Thus this is a geometric series with $r = \frac{2}{3}$ and $a = 6$. It follows that the sum is

$$\boxed{\frac{a}{1 - r} = \frac{6}{1 - \frac{2}{3}} = 18.$$

4. Use the Taylor series for $f(x) = \sin(x^3)$ to find $f^{(15)}(0)$.

We begin with the series

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots .$$

Plugging in x^3 for x , we find that

$$\sin(x^3) = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots .$$

Thus the coefficient of x^{15} is $\frac{1}{5!}$. On the other hand, the formula for Taylor series tells us that the coefficient of x^{15} is $(f^{(15)}(0))/(15!)$. Thus we have

$$\frac{1}{5!} = \frac{f^{(15)}(0)}{15!} \quad \text{or} \quad \boxed{f^{(15)}(0) = \frac{15!}{5!} = 10\,897\,286\,400.}$$

5. In this problem you will show that the series $\sum_{n=1}^{\infty} (-1)^n \ln\left(\frac{n+1}{n}\right)$ converges conditionally.

(a) Show that the following series converges: $\sum_{n=1}^{\infty} (-1)^n \ln\left(\frac{n+1}{n}\right)$.

We apply the alternating series test. We first check that the unsigned summands are decreasing; in other words, we need to check that

$$\ln\left(\frac{n+2}{n+1}\right) \leq \ln\left(\frac{n+1}{n}\right).$$

Because the natural logarithm is an increasing function, this is true precisely if

$$\frac{n+2}{n+1} \leq \frac{n+1}{n}.$$

This inequality can be seen to hold by cross multiplying to obtain

$$n^2 + 2n \leq n^2 + 2n + 1,$$

which is clearly true for all $n \geq 1$. We next check that the limit of the unsigned summands is zero. To this end we have by the continuity of the logarithm that

$$\lim_{n \rightarrow \infty} \ln\left(\frac{n+1}{n}\right) = \ln\left(\lim_{n \rightarrow \infty} \frac{n+1}{n}\right) = \ln(1) = 0.$$

It follows now from the alternating series test that this series converges.

- (b) Show that $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ diverges by finding an expression for the n th partial sum s_n .
 (Hint: Write out the first few partial sums using the identity $\ln(a/b) = \ln(a) - \ln(b)$.)

The first few partial sums are

$$\begin{aligned} s_1 &= \ln(2/1) \\ &= \ln(2) - \ln(1) = \ln(2) \\ s_2 &= \ln(3/2) + \ln(2/1) \\ &= (\ln(3) - \ln(2)) + (\ln(2) - \ln(1)) = \ln(3) \\ s_3 &= \ln(4/3) + \ln(3/2) + \ln(2/1) \\ &= (\ln(4) - \ln(3)) + (\ln(3) - \ln(2)) + (\ln(2) - \ln(1)) = \ln(4). \end{aligned}$$

From this it is clear that the n th partial sum will be

$$s_n = \ln(n+1),$$

so that

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \ln(n+1) = \infty.$$

The series therefore diverges, because the limit of the partial sums diverges.

6. Find the interval of convergence for the following series:

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n2^n} = \frac{x-2}{2} + \frac{(x-2)^2}{8} + \frac{(x-2)^3}{24} + \frac{(x-2)^4}{64} + \dots$$

We apply the ratio test, so that

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(x-2)^{n+1} n 2^n}{(x-2)^n (n+1) 2^{n+1}} \right| = \left| \frac{x-2}{2} \frac{n}{n+1} \right| \xrightarrow{n \rightarrow \infty} \left| \frac{x-2}{2} \right|.$$

Thus for convergence we require that

$$\begin{aligned} \left| \frac{x-2}{2} \right| < 1 &\Leftrightarrow -1 < \frac{x-2}{2} < 1 \\ &\Leftrightarrow -2 < x-2 < 2 \\ &\Leftrightarrow 0 < x < 4. \end{aligned}$$

We now just need to check the endpoints. When $x = 0$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges (it is the alternating harmonic series). On the other hand, if $x = 4$ we have

$$\sum_{n=1}^{\infty} \frac{2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges (it is the harmonic series). Thus the interval of convergence is

$$0 \leq x < 4.$$