

1. Consider the parametrized curve $\begin{cases} x = t^2 \\ y = t^3 - 4t \end{cases}$.

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, and determine at which points this graph has horizontal or vertical tangent.

$$\frac{dy}{dt} = 3t^2 - 4 \quad \text{and} \quad \frac{dx}{dt} = 2t \quad \Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{3t^2 - 4}{2t}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{6t(2t) - 2(3t^2 - 4)}{(2t)^2}}{2t} = \boxed{\frac{3t^2 + 4}{4t^3}}$$

For the points at which the slope is horizontal, we solve

$$\frac{dy}{dt} = 3t^2 - 4 = 0 \quad \Rightarrow \quad t^2 = \frac{4}{3},$$

so the slope is horizontal when $t = \pm \frac{2}{\sqrt{3}}$ which is at the points

$$\boxed{(x, y) = \left(\frac{4}{3}, -\frac{16\sqrt{3}}{9}\right), \left(-\frac{4}{3}, \frac{16\sqrt{3}}{9}\right)}$$

For the points at which the slope is vertical, we solve $\frac{dx}{dt} = 2t = 0$ so the slope is vertical when $t = 0$, which is at the point

$$\boxed{(x, y) = (0, 0)}$$

(b) Set up, but do not evaluate, an integral (in t) equal to the length of the arch lying above the x -axis (see the picture on the board).

We first find the limits of integration. Note that $y = 0$ implies that $t = 0, -2, 2$. When $t = 0$, we are at the origin. When $t = \pm 2$, we are at the point $(4, 0)$. The arc length is therefore given by

$$\text{Length} = \int_0^2 \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} dt = \boxed{\int_0^2 \sqrt{(3t^2 - 4)^2 + (2t)^2} dt}$$

(Note: Technically, this is finding the length of the bottom arch. The top arch is parametrized from -2 to 0 . They clearly have the same length, however, so this works, too.)

2. Consider the polar curve $r = 1 + 2 \sin \theta$.

(a) Find the slope of the line tangent to this curve when $\theta = 0$.

The slope is given by $\frac{dy}{dx}$, so we need expressions for y and x . These are given by

$$y = r \sin \theta = (1 + 2 \sin \theta) \sin \theta \quad \text{and} \quad x = r \cos \theta = (1 + 2 \sin \theta) \cos \theta.$$

Differentiating we have that

$$\frac{dy}{d\theta} = 2 \cos \theta \sin \theta + (1 + 2 \sin \theta) \cos \theta$$

and

$$\frac{dx}{d\theta} = 2 \cos^2 \theta - (1 + 2 \sin \theta) \sin \theta.$$

Plugging in $\theta = 0$ we have $\frac{dy}{d\theta} = 1$ and $\frac{dx}{d\theta} = 2$, so finally

$$\boxed{\frac{dy}{dx} = \frac{1}{2} \quad \text{at} \quad \theta = 0.}$$

(b) Set up, but do not evaluate, an integral (in θ) equal to the area inside the curve $r^2 = 8 \cos(2\theta)$ and outside the curve $r = 2$ (see the picture on the board).

We just need to find where these curves intersect, for which we have

$$8 \cos(2\theta) = 4 \quad \Rightarrow \quad \cos(2\theta) = \frac{1}{2} \quad \Rightarrow \quad 2\theta = \frac{\pi}{3} \quad \Rightarrow \quad \theta = \frac{\pi}{6}$$

Using the symmetry of the picture and the formula $A = \int \frac{1}{2} r^2 d\theta$, we then have that

$$\boxed{\text{Area} = 4 \int_0^{\pi/6} (4 \cos(2\theta) - 2) d\theta}$$

3. Consider the vectors $\mathbf{a} = \langle 2, 1, 1 \rangle$ and $\mathbf{b} = \langle -3, 0, 4 \rangle$. Find the following (angles may be left in terms of inverse trig functions if necessary):

(a) $\mathbf{a} \cdot \mathbf{b}$

$$= (2)(-3) + (1)(0) + (1)(4) = \boxed{-2}$$

(b) $\mathbf{a} \times \mathbf{b}$

$$= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -3 & 0 & 4 \end{bmatrix} = \mathbf{i}(4 - 0) - \mathbf{j}(8 - (-3)) + \mathbf{k}(0 - (-3)) = \boxed{\langle 4, -11, 3 \rangle}$$

(c) the (cosine of the) angle between \mathbf{a} and \mathbf{b}

$$= \frac{|\mathbf{a} \cdot \mathbf{b}|}{|\mathbf{a}||\mathbf{b}|} = \frac{|-2|}{\sqrt{2^2 + 1^2 + 1^2} \sqrt{(-3)^2 + 0^2 + 4^2}} = \boxed{\frac{2}{5\sqrt{6}}}$$

(d) the projection of \mathbf{b} onto \mathbf{a}

$$= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a} = \frac{-2}{6} \langle 2, 1, 1 \rangle = \boxed{\langle -\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3} \rangle}$$

(e) the area of the triangle with \mathbf{a} and \mathbf{b} as two of its sides

$$= \frac{1}{2} |\mathbf{a} \times \mathbf{b}| = \frac{1}{2} |\langle 4, -11, 3 \rangle| = \frac{1}{2} \sqrt{4^2 + (-11)^2 + 3^2} = \boxed{\frac{\sqrt{146}}{2}}$$

(f) an equation for the plane through $(1, 2, 3)$ and parallel to both \mathbf{a} and \mathbf{b}

Because the plane is parallel both to \mathbf{a} and to \mathbf{b} , the normal vector is orthogonal to both \mathbf{a} and \mathbf{b} . Thus $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 4, -11, 3 \rangle$. Thus the plane is given by

$$4x - 11y + 3z + d = 0$$

for some d . Plugging in the point $(1, 2, 3)$, we have

$$4 - 22 + 9 + d = 0 \quad \Rightarrow \quad d = 9.$$

Thus the equation is

$$\boxed{4x - 11y + 3z + 9 = 0}$$

(g) the (cosine of the) angle between the plane through $(1, 2, 3)$ with normal vector \mathbf{a} and the plane through $(1, 2, 3)$ with normal vector \mathbf{b} ;

The angle between the planes is the angle between their normals, which is the angle between \mathbf{a} and \mathbf{b} , whose cosine we have already found is

$$\boxed{\cos \theta = \frac{2}{5\sqrt{6}}}$$

(h) parametric equations for the line of intersection between the two planes in part (g);

By construction, both planes pass through the point $(1, 2, 3)$, so all we need is the direction of the line of intersection. This is perpendicular both to \mathbf{a} and to \mathbf{b} , and thus is the same as $\mathbf{a} \times \mathbf{b} = \langle 4, -11, 3 \rangle$. The vector form of the line is therefore

$$\langle 1, 2, 3 \rangle + t \langle 4, -11, 3 \rangle,$$

so the parametric form is

$$\boxed{\begin{cases} x = 1 + 4t \\ y = 2 - 11t \\ z = 3 + 3t \end{cases}}$$

(i) the volume of the parallelepiped three of whose sides are \mathbf{a} , \mathbf{b} , and $\langle 1, 2, 3 \rangle$

$$= |\langle 1, 2, 3 \rangle \cdot (\mathbf{a} \times \mathbf{b})| = |\langle 1, 2, 3 \rangle \cdot \langle 4, -11, 3 \rangle| = |4 - 22 + 9| = |-9| = \boxed{9}$$

4. Let \mathbf{a} and \mathbf{b} be the same vectors as above.

(a) Suppose \mathbf{b} represents a force vector (in Newtons) acting on an object at the point $(1, 2, 3)$, moving it in the direction of \mathbf{a} a distance of 5 meters. Find the work done by this force.

Note that from the previous problem, because $\mathbf{F} = \mathbf{b}$ and because \mathbf{D} is in the direction of \mathbf{a} , we know the cosine of the angle between \mathbf{F} and \mathbf{D} . We also know the length of \mathbf{F} , and we're told the length of \mathbf{D} . We can therefore calculate

$$W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}||\mathbf{D}| \cos \theta = |\mathbf{b}|5 \frac{2}{5\sqrt{6}} = \frac{10}{\sqrt{6}} = \boxed{\frac{5\sqrt{6}}{3}}$$

(b) Suppose there is a wrench fixed on a bolt at the origin whose handle points in the direction of the positive y axis. (The bolt is aligned so that the wrench will turn around in the xy -plane.) Suppose a force of 50 Newtons is applied at some point along this wrench at an angle of $2\pi/3$, measured counterclockwise in the xy -plane from the positive x -axis in the usual way. How far along the wrench from the bolt should this force be applied in order to supply 100 Joules of torque to the bolt?

We will use the equation $|\tau| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin \theta$, where \mathbf{r} gives the position of the point at which force is applied, \mathbf{F} is the force, and θ is the angle between the two vectors. We're told that $|\mathbf{F}| = 50$. We want to find $|\mathbf{r}|$, so all we need is the angle θ . The wrench goes along the positive y -axis (at an angle of $\pi/2$), and the force is applied at an angle of $2\pi/3$. Thus the angle between the two is $\pi/6$. We therefore have

$$|\tau| = |\mathbf{r}||\mathbf{F}| \sin \theta = |\mathbf{r}|50 \frac{1}{2} = 25|\mathbf{r}|.$$

If this is to equal 100, we must have

$$\boxed{|\mathbf{r}| = 4}$$