Quiz 1 Solutions    June 19, 2006

1. Let $S = \{-t+5, t-3\}$ and $T = \{t-1, t+1\}$ be ordered bases for $P_1$, and let $\mathbf{v} = 3t-1$.

(a) Find $[\mathbf{v}]_T$ directly.

We solve $3t-1 = a(t-1)+b(t+1) = (a+b)t+(-a+b)$, which has solution $a=2$, $b=1$. Thus we have $[\mathbf{v}]_T = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(b) Find the transition matrix $P$ from $T$ to $S$.

The columns of $P$ are the $S$-coordinates of the corresponding $T$-vectors. Because $t-1 = 1(-t+5)+2(t-3)$, we have that $[t-1]_S = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Similarly we have $[t+1]_S = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, so that $P = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

(c) Use parts (a) and (b) to find $[\mathbf{v}]_S$.

We know that $[\mathbf{v}]_S = P[\mathbf{v}]_T$, so we have

$$[3t-1]_S = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$$

2. Let $W$ be the subspace of $M_{22}$ consisting of all symmetric $2\times 2$ matrices. Thus $W$ is all matrices of the form $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ for some real numbers $a, b, c$.

(a) Find a finite set of matrices in $W$ that spans $W$.

Every matrix in $W$ is of the form

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

It follows that these three matrices span $W$.

(b) Verify that the following set of vectors in $W$ is linearly independent:

$$\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

We need to show that the following equation has only one solution:

$$a \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This leads to the following augmented matrix, which reduces as shown:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The corresponding system of linear equations has precisely one solution (back substituting, we never come upon an equation with more than one variable more than the one below). It follows that the set is linearly independent.

(c) What dimension is $W$? Explain (based on parts (a) and (b) above).

The dimension of $W$ is three. The fact that three vectors can span $W$ means that the dimension is no more than three. The fact that there are three linearly independent vectors in $W$ means the dimension is no less than three.

3. For each of the following seven statements, indicate clearly whether it is true or false. **For all problems, assume that $V$ is a *finite-dimensional* vector space.** For TWO of the statements, also do the following (in the space below): If the statement is true, then explain why. If the statement is false, then provide a counterexample.

(a) ______ For any vector $\mathbf{v}$ in $V$, it is always true that $0 \odot \mathbf{v} = \mathbf{0}$.

True. This is part of theorem 3.2, and follows directly from the eight defining properties of a vector space.

(b) ______ A set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of linearly independent vectors in $V$ is necessarily a basis for the subspace they span.

True. By definition, they span their span, and they are also linearly independent by assumption. Thus they form a basis for their span.

(c) ______ If $S_1$ is a linearly independent set of vectors in $V$, then any set $S_2$ containing $S_1$ is also linearly independent.

False. For example, take $S_2$ to be the vectors in $S_1$ along with the zero vector.

(d) ______ If $\dim(V) = n$, then any set of $n$ or more vectors must span $V$.

False. For example, one could take $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ in $\mathbf{R}^2$.

(e) ______ Any two bases for $V$ must contain the same number of vectors.

True. This is corollary 3.1, and is why (finite) dimension is well-defined.

(f) ______ Every set that spans $V$ contains a basis for that vector space.

True. This is theorem 3.8. One obtains a basis by throwing away dependent vectors one at a time.

(g) ______ Suppose $W$ is a subspace of $V$ and that $S$ is a basis for $V$. Then some subset of $S$ is a basis for $W$.

False. For example, take $S$ to be the standard basis for $V = \mathbf{R}^2$, and take $W$ to be spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. No single vector in $S$ even lies in $W$, so no collection of them can span it.