

## Solutions to review problems for test one

### Chapter Three

#### **T/F**

2. True; this was part of the first theorem we proved about vector spaces.
4. True; this is exactly the subspace test.
5. True; this is the definition.
6. False; the zero vector is a subspace by itself; all other subspaces contain infinitely many vectors, however.
7. True; this is the nullspace of the matrix (or the kernel of the corresponding linear map).
8. True; this is the definition of span.
9. True; dependence means one is a linear combination of all the others; in this case, there is only one other, so linear combination becomes scalar multiple.
10. False; a subset of a linearly independent set is still linearly independent.
11. False; any set containing the zero vector is linearly dependent.
12. True; this is the definition.
13. False; dimension counts basis vectors, not total number of all vectors.
14. True; this is a theorem, and makes (finite) dimension well-defined.
15. True; this is a theorem.
16. True; this is a theorem.
18. True; this is exactly what coordinates are.
19. False; rank is the number of nonzero rows in the row-echelon form.
20. False; if a square matrix has full rank, then solutions are unique (the linear map it represents is an isomorphism; in particular it is 1-1, so solutions are unique).
21. True; if a square matrix has full rank, it is non-singular; otherwise its rank is smaller.

#### **Quiz**

1. No; addition is not commutative, for example.
2. No; it fails property 8, for example.
3. Yes; a spanning set is obtained by pulling out constants:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

This is then easily seen to be linearly independent (they aren't multiples of one another), and so it is a basis.

4. No; it's not closed under scalar multiplication (by a negative number).

5. It is a subspace (check directly). In general, we have  $p(t) = at^2 + bt + c$ . If we also have that  $p(0) = 0$ , then this means that  $c = 0$ . Thus we are considering the subset of  $P_2$  consisting of polynomials of the form  $at^2 + bt$ . We find a spanning set by pulling out constants, obtaining  $\{t^2, t\}$ . This is linearly independent, and so is a basis.

6. Yes; this can be checked in many different ways. First, line them up in a matrix. You can check that the linear map it represents is onto, or that the rank equals the number of rows, or that when row reduced there are no zero rows. Because the matrix is square, you can also check that the linear map it represents is 1-1, that the rank equals the number of columns, that all columns have leading terms when row reduced, or that the matrix is nonsingular.

7. Line them up and row reduce. The columns that contain leading terms form a basis for the set they span. Because there are three such, the dimension of the space they span is three, and so must be all of  $\mathbf{R}^3$ .

8. Row reduce and write down a general solution to the homogeneous equation. This matrix reduces to

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The homogeneous system this corresponds to is

$$a + 2b + c + d = 0, \quad b + d = 0,$$

which has general solution as follows:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} r - s \\ -r \\ s \\ r \end{bmatrix} = r \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

These last two vectors form a basis for the null space.

9. The first two rows are a basis for the row space. If you want, you can interchange rows and columns. Then we're looking for a basis for the column space, which is found by row-reducing and circling the columns that contain leading terms. You find that the first two columns span the column space of your matrix, so the first two rows span the row space of the given matrix.

10. This is just rank-nullity.

## T/F

1. True; follows directly from the definition of linearity and the fact that subtraction is just addition of a negative.
2. False; there are lots of nonzero vectors that map to the zero vector.
3. True; this follows from the fact that the kernel is a subspace of the domain.
4. False; this is the definition of onto, and not all linear maps are onto.
5. False; the rank-nullity theorem requires that the dimension of the kernel plus the dimension of the image add up to the dimension of the domain, which in this case is four.
6. True; this is a theorem.
7. True; this is a theorem.
8. True; this is the definition.
9. True; by rank-nullity, the fact that  $L$  is onto, along with the fact that the range and domain have the same dimension, implies that  $L$  is also 1-1, and hence is an isomorphism.
10. False; the map that sends every vector to the zero vector is linear.

## Quiz

1. Yes; check directly.
2. (a) Check directly; (b)  $A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$  (apply the map to the standard basis, then line up the results).
3. (a) Any matrix whose third column is negative its first column will work; (b) No, it has nontrivial kernel.

4. We first write  $\begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ , and find that  $a = 2$ ,  $b = 1$ ,  $c = -3$ .

Thus

$$\begin{aligned} L\left(\begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}\right) &= L\left(2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right) \\ &= 2L\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) + 1L\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) - 3L\left(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right) = 2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 4 \end{bmatrix}. \end{aligned}$$

5. We apply  $L$  to the  $T$ -basis vectors first, obtaining

$$L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad L\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}.$$

We then find  $S$ -coordinates for these, but because  $S$  is the standard basis, these vectors are their own  $S$ -coordinates. Finally we line up the results in a matrix, obtaining

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}.$$

6. (a) We apply  $L$  to the  $S$ -basis vectors and take the  $T$ -coordinates of the results. Because these bases are both standard, this is straightforward, and we obtain

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}.$$

(b) We find the  $S$ -coordinates for the  $S'$ -basis vectors. Because  $S$  is the standard basis,  $S$ -coordinates are trivial. Thus we have

$$P = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

(c) As before, we have

$$Q = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

(d) We could find this directly as in (a), by applying  $L$  to each of the  $S'$ -basis vectors, finding the  $T'$ -coordinates of the results, and then lining them up. Or we can simply find  $Q^{-1}$  and calculate  $B = Q^{-1}AP$ . To this end we note that

$$Q^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}.$$

Multiplying, we obtain

$$B = \begin{bmatrix} -1 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}.$$