Test 1 Solutions July 5, 2006

1. Let $L : P_2 \to P_2$ be defined by $L(at^2+bt+c) = (a+2b-c)t^2+(a+c)t+(4a-4b+5c)$.

(a) Find the matrix representation for L with respect to the basis

$$S = \{t^2,\ t^2-t-2,\ 2t^2-t-4\}.$$

The columns are going to be the S-coordinates of L of the S-basis vectors. Thus the first column is $[L(t^2)]_S = [t^2+t+4]_S$. The S-coordinate vector of t^2+t+4 is $\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}$, where α, β, γ solve the equation

$$t^2+t+4 = \alpha(t^2)+\beta(t^2-t-2)+\gamma(2t^2-t-4) = (\alpha+\beta+2\gamma)t^2+(-\beta-\gamma)t+(-2\beta-4\gamma).$$

To solve this, we reduce the corresponding coefficient matrix as follows:

$$\begin{bmatrix}1 & 1 & 2 & | & 1\\ 0 & -1 & -1 & | & 1\\ 0 & -2 & -4 & | & 4\end{bmatrix} \mapsto \begin{bmatrix}1 & 1 & 2 & | & 1\\ 0 & 1 & 1 & | & -1\\ 0 & 0 & 1 & | & -1\end{bmatrix},$$

which has solution $\gamma = -1,\ \beta = 0,\ \alpha = 3$. Thus the first column of the matrix representation is

$$[L(t^2)]_S = [t^2+t+4]_S = \begin{bmatrix}3\\ 0\\ 1\end{bmatrix}.$$

For the second column, we do the same thing with the second S-basis vector. Thus we have that the second column is

$$[L(t^2-t-2)]_S = [t^2-t-2]_S = \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}.$$

(The coordinates were easy to find here because the output was exactly a basis vector already.)

For the third column, we have

$$[L(2t^2-t-4)]_S = [4t^2-2t-8]_S = \begin{bmatrix}0\\ 0\\ 2\end{bmatrix},$$

where, again, the coordinates were easy to find because the output was just twice a basis vector.

Thus the matrix representation for L with respect to the basis S is

$$\begin{bmatrix}3 & 0 & 0\\ 0 & 1 & 0\\ -1 & 0 & 2\end{bmatrix}.$$

(b) The transition matrix $P_{T \leftarrow S}$ from S to another basis T is given by

$$P_{T \leftarrow S} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}.$$

Find the transition matrix from T to S by inverting this matrix.

`We invert the given matrix as follows:`

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\mapsto \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right]$$

$$\mapsto \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right].$$

`So we find that`

$$P_{T \leftarrow S}^{-1} = P_{S \leftarrow T} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}.$$

(c) Use these matrices to find the matrix representation for L with respect to T.

`If` A `is the matrix we found in part (a), then we have that the matrix representation for` L `with respect to the basis` T `is given by`

$$P_{T \leftarrow S} A P_{S \leftarrow T} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ -4 & 4 & -2 \\ -2 & 1 & 1 \end{bmatrix}.$$

2. Suppose $S = \{t+1, t-1, t^2+1\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ are ordered bases for P_2, and suppose the transition matrix from T to S is $P_{S \leftarrow T} = \begin{bmatrix} 2 & 4 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Suppose $\mathbf{v}$ is a vector in P_2 with T-coordinates $[\mathbf{v}]_T = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. What polynomial does $\mathbf{v}$ represent?

Method I: Use the transition matrix as it's intended to be used. We know that $P_{S \leftarrow T}[\mathbf{v}]_T = [\mathbf{v}]_S$, so we have that

$$[\mathbf{v}]_S = \begin{bmatrix} 2 & 4 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

Because these are S-coordinates, it follows that

$$\mathbf{v} = -1(t+1) + 2(t-1) + 0(t^2+1) = t - 3.$$

Method II: Use the transition matrix to find the basis T explicitly. The columns of the transition matrix are the S-coordinates of the T-basis. Since we know the S-basis, we can find the T-basis as follows:

$$\begin{aligned} \mathbf{w}_1 &= 2(t+1) + 1(t-1) + 0(t^2+1) = 3t + 1, \\ \mathbf{w}_2 &= 4(t+1) - 1(t-1) + 1(t^2+1) = t^2 + 3t + 6, \\ \mathbf{w}_3 &= 1(t+1) + 0(t-1) + 1(t^2+1) = t^2 + t + 2. \end{aligned}$$

Then from the given T-coordinates of $\mathbf{v}$ we find that

$$\mathbf{v} = 1(3t+1) - 1(t^2+3t+6) + 1(t^2+t+2) = t - 3.$$

3. Consider the linear map $L : \mathbf{R}^4 \to \mathbf{R}^4$ given by

$$L\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a + 2b + c + d \\ b + d \\ 2a + 3b + 2c + d \\ a + c - d \end{bmatrix}.$$

(a) Find the standard matrix representation for L.

The columns of the standard matrix representation are just the images of the standard basis vectors (technically, they're the S-coordinates of the images, just like in problem #1; but because we're using the standard basis, the vectors all equal their own coordinates). Thus we have

$$L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \end{bmatrix},$$

$$L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad L\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

So the transition matrix is

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 2 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix}.$$

(b) Verify the rank-nullity theorem for this map by finding bases for the image and for the kernel.

We row-reduce the previous matrix as follows:

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 2 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

To find a basis for the image, we note that the leading terms are in columns one and three, so the image is spanned by the original first and third column vectors. In other words,

$$\mathtt{Im}(L) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

Because these vectors are not multiples of one another, they are linearly independent. Thus they form a basis, and so the dimension of the image is two.

For the kernel, we find a general solution to the homogeneous system of equations corresponding to the matrix, which is

$$a + 2b + c + d = 0, \qquad b + d = 0.$$

From this we find that $d = r$, $c = s$, $b = -r$, $a = 2r - s - r = r - s$ is a general solution. It follows that the kernel is

$$\mathtt{Ker}(L) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Because these vectors are not multiples of one another, they are linearly independent. Thus they form a basis, and so the dimension of the kernel is two.

To verify the rank-nullity theorem, we note that the dimension of the domain $\mathbf{R}^4$ is four, and $2+2=4$.

4. For each of the following ten statements, indicate clearly whether it is true or false. For TWO of the statements, also do the following (in the space at the bottom

of the page): If the statement is true, then explain why. If the statement is false, then provide a counterexample.

(a) ______ The set of all continuous functions f so that $\int_0^1 f(x)\,dx = 0$ is a subspace of $C(-\infty, \infty)$.

`True; if` $\int_0^1 f(x)\,dx = 0$ `and` $\int_0^1 g(x)\,dx = 0$, `then` $\int_0^1 (f+g)(x)\,dx = 0$ `and` $\int_0^1 (cf(x))\,dx = 0$.

(b) ______ The rank of a matrix is equal to the number of nonzero rows in its row-reduced form.

`True; the number of nonzero rows is the same as the number of leading terms, which is the number of vectors in a basis for the image, which is the same as the rank.`

(c) ______ If $L : V \to W$ is a linear map, then the image of a linearly independent set of vectors in V is a linearly independent set of vectors in W.

`False; take any nontrivial vector space and let` L `be the linear operator that sends every vector to the zero vector, for example.`

(d) ______ Transition matrices are always invertible.

`True; the inverse is the opposite transition.`

(e) ______ If $\dim(V) = n$, then any set with fewer than n vectors must be linearly independent in V.

`False; pick a vector and a multiple of it, for example (even better, just pick the zero vector).`

(f) ______ A matrix is invertible if and only if it row-reduces to the identity matrix.

`True; this follows from the procedure we learned for finding the inverse.`

(g) ______ Similar matrices have the same rank.

`True; they represent the same linear map with respect to different bases, so the rank of the matrices is equal to the dimension of the image of the map.`

(h) ______ If a linear map is one-to-one, then its matrix representation cannot contain any zero rows.

`False; every column must contain a leading term, but the matrix could have more rows than columns.`

(i) ______ The integers are a subspace of $\mathbf{R}$.

`False; they are not closed under scalar multiplication.`

(j) ______ If $L : V \to V$ is a one-to-one linear operator, then L must be onto.

True; one-to-one means the kernel has dimension zero, so rank-nullity says the dimension of the image equals the dimension of the domain; but the domain and range are the same thing, so the image is a subspace of V with the same dimension as V; it follows that the image is all of V.