Solutions to Worksheet 1

1. Let V be the set of all 2×1 matrices $\begin{bmatrix} x \\ y \end{bmatrix}$ with operations defined as follows:

$$\begin{bmatrix} p \\ q \end{bmatrix} \oplus \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} p+s \\ q+t \end{bmatrix} \qquad c \odot \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} cs \\ 0 \end{bmatrix}.$$

Is V a vector space? If so, prove it. If not, list any properties of Definition 3.4 that fail to hold.

No, because property (8) fails. For example, if $\mathbf{u}$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, property (8) says that we should have $1 \odot \mathbf{u} = \mathbf{u}$, but instead we have $1 \odot \mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \mathbf{u}$.

2. Let $\mathbf{Z}$ denote the set of all integers $\ldots - 2, -1, 0, 1, 2, \ldots$. Is $\mathbf{Z}$ a subspace of the set of all real numbers $\mathbf{R}$ (with the usual operations)? If so, prove it. If not, why not?

No, because it is not closed under scalar multiplication. For example, no matter what integer $\mathbf{u}$ is, $\pi \odot \mathbf{u}$ is not going to be an integer.

3. Let $V = C(-1, 1)$ be the set of all real-valued functions that are continuous between -1 and 1 (but not necessarily at the endpoints). Let W be the subset of V consisting of those functions with the further property that

$$\int_{-1}^{1} f(x)\, dx = 0.$$

Is W a subspace of V? If so, prove it. If not, why not?

Yes. To see this, let f and g be functions in W. So we know that $\int_{-1}^{1} f\, dx = 0 = \int_{-1}^{1} g\, dx$. To see that $f+g$ is in W we need to show that $\int_{-1}^{1} (f+g)\, dx = 0$. But this is true because, by basic properties of integrals, we have that $\int_{-1}^{1} (f+g)\, dx = \int_{-1}^{1} f\, dx + \int_{-1}^{1} g\, dx = 0 + 0 = 0$.

For closure under scalar multiplication, we again start with f in W (so $\int_{-1}^{1} f\, dx = 0$), and we need to show that for any real number c we have that $\int_{-1}^{1} (cf)\, dx = 0$. But this is true because $\int_{-1}^{1} (cf)\, dx = c \int_{-1}^{1} f\, dx = c(0) = 0$.

4. (a) Let $S = \{t^2 + 2t + 1,\ t^2 + 3,\ t - 1\}$ be a set of vectors in P_2. Is $t^2 - 1$ in the span of S?

No. Saying that $t^2 - 1$ is in the span of S is equivalent to saying that the following equation has a solution:

$$\alpha(t^2 + 2t + 1) + \beta(t^2 + 3) + \gamma(t - 1) = t^2 - 1.$$

This equation turns into

$$(\alpha + \beta)t^2 + (2\alpha + \gamma)t + (\alpha + 3\beta - \gamma) = t^2 - 1,$$

which implies, by equating corresponding coefficients, that

$$\alpha+\beta=1, \qquad 2\alpha+\gamma=0, \qquad \alpha+3\beta-\gamma=-1.$$

This system of linear equations turns into the following augmented matrix, which row reduces as shown:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 3 & -1 & -1 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 0 & -4 \end{array}\right].$$

This corresponds to a system with no solution (the last row corresponds to the equation $0=-4$). Thus the system has no solution, so t^2-1 is not in the span.

(b) Find a relationship that must be satisfied by a, b, c in order for at^2+bt+c to lie in the span of S.

To test whether a random vector at^2+bt+c lies in the span of S, we would set up exactly the same system of equations as above, except that the rightmost column would be $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$. The last row then row reduces to $\left[\begin{array}{ccc|c} 0 & 0 & 0 & -3a+b+c \end{array}\right]$. Thus the system has a solution precisely when $-3a+b+c=0$.

5. (a) Show that the set

$$S=\left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right\}$$

spans M_{22}.

We need to show that the following equation always has a solution:

$$\alpha\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}+\beta\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}+\gamma\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\delta\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Multiplying out the left-hand side and equating corresponding entries, we obtain the following system of four equations:

$$\alpha+\gamma=a, \qquad \alpha+\delta=b, \qquad \beta+\delta=c, \qquad \beta+\gamma+\delta=d.$$

This corresponds to the following augmented matrix, which row reduces as shown:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & a \\ 1 & 0 & 0 & 1 & b \\ 0 & 1 & 0 & 1 & c \\ 0 & 1 & 1 & 1 & d \end{array}\right] \mapsto \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & a \\ 0 & 1 & 0 & 1 & c \\ 0 & 0 & 1 & 0 & d-c \\ 0 & 0 & 0 & 1 & b-a+d-c \end{array}\right].$$

The corresponding system has explicit solution $\alpha=a-d+c$, $\beta=2c-b+a-d$, $\gamma=d-c$, and $\delta=b-a+d-c$.

(b) Write $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ as a linear combination of the matrices in S.

Using the solution above (with $a = 1$, $b = 2$, $c = 3$, and $d = 4$) we find that

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 0 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$