Week 2 worksheet
(due Monday, June 19 at the beginning of class)

1. (a) Show that the following set of vectors in M_{22} is linearly dependent:

$$\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}, \begin{bmatrix}0 & 2\\1 & -1\end{bmatrix}, \begin{bmatrix}3 & -4\\1 & 5\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}.$$

We need to show that the following equation has a nontrivial solution:

$$a\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix} + b\begin{bmatrix}0 & 2\\1 & -1\end{bmatrix} + c\begin{bmatrix}3 & -4\\1 & 5\end{bmatrix} + d\begin{bmatrix}1 & 1\\0 & 2\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}.$$

Combining the left-hand side and setting corresponding entries equal, we obtain the following system of equations:

$$a + 3c + d = 0, \quad 2b - 4c + d = 0, \quad a + b + c = 0, \quad a - b + 5c + 2d = 0,$$

which in turn corresponds to the following augmented matrix, which reduces as shown:

$$\left[\begin{array}{cccc|c}1 & 0 & 3 & 1 & 0\\0 & 2 & -4 & 1 & 0\\1 & 1 & 1 & 0 & 0\\1 & -1 & 5 & 2 & 0\end{array}\right] \mapsto \left[\begin{array}{cccc|c}1 & 0 & 3 & 1 & 0\\0 & 2 & -4 & 1 & 0\\0 & 0 & 0 & 3 & 0\\0 & 0 & 0 & 0 & 0\end{array}\right].$$

Because the second row contains two more unknowns than the third, there are infinitely many solutions (when back-substituting, we get to assign c to be any real number before solving for b). It follows that the system is linearly dependent.

(b) Express one vector as a linear combination of the others.

Explicitly solving the system above, we find that $d = 0,\ c = r,\ b = 2c = 2r,\ a = -3c = -3r$. We obtain a particular solution by choosing a value for r, say $r = 1$, so that $a = -3,\ b = 2,\ c = 1,\ d = 0$. Thus we have the equation

$$-3\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix} + 2\begin{bmatrix}0 & 2\\1 & -1\end{bmatrix} + 1\begin{bmatrix}3 & -4\\1 & 5\end{bmatrix} + 0\begin{bmatrix}1 & 1\\0 & 2\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}.$$

To express one matrix in terms of the others, we move any of the first three matrices to the other side of the equality and solve.

2. Find a basis for $\mathbf{R}^4$ containing the following vectors:

$$\begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\-1\end{bmatrix}.$$

`We adjoin the standard basis to obtain a spanning set, and then row reduce to find a basis inside that set. Thus we row reduce as follows:`

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 \end{bmatrix}.$$

`Because the initial nonzero terms lie in the first, second, third, and fifth columns, the correponding vectors form a basis. Thus the basis is`

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \right\}.$$

3. Find a spanning set for the subspace of P_2 consisting of polynomials $at^2 + bt + c$, where $a + 2b = c$.

`A general vector in` W `looks like` $at^2 + bt + (a + 2b) = a(t^2 + 1) + b(t + 2)$. `It follows that` $\{t^2 + 1,\ t + 2\}$ `is a spanning set for` W.

4. Let $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ and $T = \left\{ \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \end{bmatrix} \right\}$ be ordered bases for $\mathbf{R}^2$. Let $\mathbf{v} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$.

(a) Find the coordinate vector for $\mathbf{v}$ with respect to the basis T.

`We write` $\begin{bmatrix} 4 \\ 4 \end{bmatrix} = a \begin{bmatrix} 4 \\ 2 \end{bmatrix} + b \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ `and solve for` a `and` b`, finding that` $a = 2$ `and` $b = -1$. `Thus we have` $[\mathbf{v}]_T = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

(b) Find the transition matrix P from the T-basis to the S-basis.

`We first express each vector in the` T`-basis in terms of the` S`-basis. This comes down to solving two systems with the following corresponding augmented matrices:`

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & -1 & 2 \end{array}\right], \quad \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & -1 & 0 \end{array}\right].$$

`We find that the` S`-coordinates for` $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ `are` $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$`, while those of` $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$ `are` $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$. `Thus we have` $P = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$.

(c) Use your answer to (b) to find the coordinate vector for $\mathbf{v}$ with respect to S.

`Using the fact that` $[\mathbf{v}]_S = P[\mathbf{v}]_T$, `we find that`

$$[\mathbf{v}]_S = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

5. (a) Let $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $T = \{t+2, 1\}$ be ordered bases for P_1. Find the basis S using the fact that the transition matrix from S to T is $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

`The first column of the transition matrix is, by definition, the` T`-coordinate vector for the first of the` S `vectors. In other words,`

$$\mathbf{v}_1 = 2(t+2) + 1(1) = 2t + 5.$$

`We similarly find that` $\mathbf{v}_2 = t + 3$.

(b) Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2\}$ be ordered bases for $\mathbf{R}^2$. Find the basis T, using the fact that the transition matrix from S to T is $\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$.

`Just as above, the first column of the transition matrix is, by definition, the` T`-coordinate vector for the first of the` S `vectors. In other words,`

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\mathbf{w}_1 - 1\mathbf{w}_2.$$

`Similarly, from the second column we obtain`

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 3\mathbf{w}_1 + 2\mathbf{w}_2.$$

`If we let` $\mathbf{w}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ `and` $\mathbf{w}_2 = \begin{bmatrix} c \\ d \end{bmatrix}$, `we arrive at the following system of four equations in four unknowns (actually two pairs of systems, each of two equations in two unknowns):`

$$\begin{bmatrix} 2 & 0 & -1 & 0 & | & 1 \\ 0 & 2 & 0 & -1 & | & 0 \\ 3 & 0 & 2 & 0 & | & 1 \\ 0 & 3 & 0 & 2 & | & -1 \end{bmatrix}$$

`the solution to which implies that` $\mathbf{w}_1 = \begin{bmatrix} 3/7 \\ -1/7 \end{bmatrix}$ `and` $\mathbf{w}_2 = \begin{bmatrix} -1/7 \\ -2/7 \end{bmatrix}$.