

Week 3 worksheet solutions

1. Let $L : P_1 \rightarrow P_2$ be a linear transformation for which we know that

$$L(t+1) = t^2 + t - 2 \quad \text{and} \quad L(t-1) = t^2 - t - 1.$$

Find $L(6t-4)$.

Note that $\{t+1, t-1\}$ is a basis for P_1 , and that

$$6t-4 = 1(t+1) + 5(t-1).$$

It follows that

$$\begin{aligned} L(6t-4) &= L((t+1) + 5(t-1)) = L(t+1) + 5L(t-1) \\ &= (t^2 + t - 2) + 5(t^2 - t - 1) = 6t^2 - 4t - 7. \end{aligned}$$

2. (a) Find the standard matrix representing the linear transformation $L : \mathbf{R}^3 \rightarrow \mathbf{R}^4$ where

$$L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a-b \\ a+c \\ b+c \\ b-a \end{bmatrix}.$$

We apply the map to the standard basis vectors, obtaining

$$L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

It follows that the standard matrix representation for L is given by

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

(b) Find a set of vectors spanning the kernel of L .

We solve the matrix equation $Ax = \mathbf{0}$, which turns into the following augmented matrix and reduces as shown:

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{array} \right] \mapsto \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

This corresponds to the system $a - b = 0$, $b + c = 0$, which has general solution

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -r \\ -r \\ r \end{bmatrix} = r \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

This last vector, therefore, spans the kernel of L .

(c) Find a set of vectors spanning the image of L .

Because the linear map is just matrix multiplication, we may use the fact that the columns of A span its image. Thus a spanning set is given by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

3. Define a linear transformation $L : M_{22} \rightarrow P_2$ by

$$L \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a + b + d)t^2 + (b - c)t + (c - d).$$

(a) Find a set of vectors spanning the kernel of L .

If the output of L is zero, then the coefficients are all zero. This leads to the linear system

$$a + b + d = 0, \quad b - c = 0, \quad c - d = 0,$$

which has general solution

$$d = r, \quad c = r, \quad b = r, \quad a = -2r$$

Thus the kernel is

$$\left\{ \begin{bmatrix} -2r & r \\ r & r \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

(b) Find a set of vectors spanning the image of L .

A typical image vector is of the form

$$(a + b + d)t^2 + (b - c)t + (c - d) = a(t^2) + b(t^2 + t) + c(-t + 1) + d(t^2 - 1),$$

so that a spanning set for the image is given by

$$\{t^2, t^2 + 1, 1 - t, t^2 - 1\}.$$

4. Verify the matrix version of the rank-nullity theorem for the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 & 2 \\ 2 & 4 & 3 & 3 & 3 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix}.$$

We need to find the nullity of A (i.e., the dimension of the kernel of the corresponding linear map) and the rank of A (i.e., the dimension of the image of the corresponding linear map) and make sure they add up to five (the number of columns of A , or the dimension of the domain of the corresponding linear map).

Just as in problem #2, the columns of A form a spanning set for the image. To find dimension, we need to reduce this spanning set to a basis. To this end we row reduce the matrix A down to

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first, third, and fifth columns contain the leading terms, so the first, third, and fifth vectors form a basis. In particular, the dimension of the image is three.

For the kernel, we again row reduce A (but with a column of zeros adjoined to it) just as before. The system we end up with is

$$a + 2b + c + 2d + e = 0, \quad c - d + e = 0, \quad e = 0.$$

Thus the kernel is

$$\left\{ \begin{bmatrix} -2s - 3r \\ s \\ r \\ r \\ 0 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Normally, we would again have to find a basis inside this spanning set, but, because it's only two vectors, it's easy to see that they are linearly independent (they are not multiples of one another). Thus this is a basis. In particular the dimension of the kernel is two. Three plus two is five, as required.

5. Let $L : V \rightarrow \mathbf{R}^5$ be a linear transformation.

(a) If L is one-to-one and $\dim(\text{Im } L) = 3$, what is $\dim(V)$?

Three (one-to-one implies no dimensions die, so ending with three implies we began with three).

(b) If $\dim(V) = 3$ and $\dim(\ker L) = 1$, what is $\dim(\operatorname{Im} L)$?

Two (we begin with three dimensions, and one dies, so we're left with two).

(c) If L is onto, what can we say about $\dim(V)$?

$\dim(V) \geq 5$ (the number of dimensions that survive is five, so there must have been at least five to start with).

(d) If L is one-to-one, what can we say about $\dim(V)$?

$\dim(V) \leq 5$ (no dimensions die, and yet the image fits inside \mathbf{R}^5 , so there must have been no more than five to start with).

(e) If L is one-to-one and onto, what can we say about $\dim(V)$?

$\dim(V) = 5$ (from parts (c) and (d)).