Week 5 worksheet solutions

1. Find the determinants of the following matrices (show your work):

$$\begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix} \quad \begin{bmatrix} 2 & 5 & -2 \\ 3 & 1 & 2 \\ -1 & -2 & 4 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 4 & 2 & -1 & 0 \\ 3 & 7 & 2 & 4 \end{bmatrix} \quad \begin{bmatrix} 2 & 9 & -5 & 3 & 9 \\ 3 & 1 & 5 & -3 & 1 \\ 0 & -4 & 1 & 2 & -4 \\ 8 & 5 & 6 & 9 & 5 \\ 3 & -2 & -1 & -3 & -2 \end{bmatrix}$$

(a) $(2)(3) - (-1)(4) = 10$

(b) $2[(1)(4)-(-2)(2)]-5[(3)(4)-(-1)(2)]-2[(3)(-2)-(-1)(1)] = -44$

(c) $(3)(-2)(-1)(4) = 24$ (product of diagonal entries)

(d) 0 (because two columns are equal)

2. Find all values λ for which the matrix equation $B\mathbf{x} = \mathbf{0}$ has a nontrivial solution, where

$$B = \begin{bmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{bmatrix}.$$

A theorem says that a matrix equation $B\mathbf{x} = \mathbf{0}$ has a nontrivial solution precisely when $\det(B) = 0$. Now

$$\det(B) = (1-\lambda)(4-\lambda) - (-2)(1) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3).$$

This polynomial is equal to zero precisely when $\lambda = 2$ or $\lambda = 3$, so this is also when the equation has a nontrivial solution.

3. Use determinants and cofactors to find the inverse of $\begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

The cofactors are as follows:

$$A_{11} = 2 - 3 = -1, \qquad A_{12} = -(1-0) = -1, \qquad A_{13} = 1 - 0 = 1,$$
$$A_{21} = -(3-2) = -1, \qquad A_{22} = 1 - 0 = 1, \qquad A_{23} = -(1-0) = -1,$$
$$A_{31} = 9 - 4 = 5, \qquad A_{32} = -(3-2) = -1, \qquad A_{33} = 2 - 3 = -1.$$

Multiplying by the corresponding entries and transposing gives

$$\begin{bmatrix} 1(-1) & 3(-1) & 2(1) \\ 1(-1) & 2(1) & 3(-1) \\ 0(5) & 1(-1) & 1(-1) \end{bmatrix}^T = \begin{bmatrix} -1 & -1 & 0 \\ -3 & 2 & -1 \\ 2 & -3 & -1 \end{bmatrix}.$$

Lastly, the determinant of the original matrix is -2, so we have that the inverse is

$$\begin{bmatrix} 1/2 & 1/2 & 0 \\ 3/2 & -1 & 1/2 \\ -1 & 3/2 & 1/2 \end{bmatrix}.$$

4. Find the linearization of the spherical change of coordinates map:

$$f(\rho, \phi, \theta) = (\rho \cos\theta \sin\phi,\ \ \rho \sin\theta \sin\phi,\ \ \rho\cos\phi).$$

$$J = \begin{bmatrix} \cos\theta\sin\phi & \rho\cos\theta\cos\phi & -\rho\sin\theta\sin\phi \\ \sin\theta\sin\phi & \rho\sin\theta\cos\phi & \rho\cos\theta\sin\phi \\ \cos\phi & -\rho\sin\phi & 0 \end{bmatrix}$$

5. Find the factor by which the linear map you found in #4 scales volume.

$$\begin{aligned} \texttt{det}(J) &= \cos\theta\sin\phi(\rho^2\cos\theta\sin^2\phi) - \rho\cos\theta\cos\phi(-\rho\cos\theta\sin\phi\cos\phi) \\ &\quad -\rho\sin\theta\sin\phi(-\rho\sin\theta\sin^2\phi - \rho\sin\theta\cos^2\phi) \\ &= \rho^2\sin\phi\big(\cos^2\theta\sin^2\phi + \cos^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi\big) \\ &= \rho^2\sin\phi\big(\cos^2\theta(\sin^2\phi + \cos^2\phi) + \sin^2\theta(\sin^2\phi + \cos^2\phi)\big) \\ &= \rho^2\sin\phi(\cos^2\theta + \sin^2\theta) = \rho^2\sin\phi. \end{aligned}$$