

Week 6 worksheet solutions

1. Find the eigenvalues and corresponding eigenvectors for the matrix  $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ .

We calculate

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda) - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1).$$

So the eigenvalues are  $\lambda = 3, -1$ . To find the eigenvectors, we solve as follows. Plugging in  $\lambda = 3$ , we have

$$\left[ \begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array} \right] \mapsto \left[ \begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right].$$

Thus a general solution is

$$\begin{bmatrix} r \\ 2r \end{bmatrix} = r \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

For  $\lambda = -1$  we have

$$\left[ \begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 0 \end{array} \right] \mapsto \left[ \begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right].$$

The general solution is

$$\begin{bmatrix} r \\ -2r \end{bmatrix} = r \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

2. The matrix  $A$  below has  $\lambda = 2$  as an eigenvalue of multiplicity two. Determine if this eigenvalue is defective or not.

$$A = \begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 2 & 1 & -1 \\ 2 & 3 & 4 & 1 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

Plugging in the eigenvalue  $\lambda = 2$ , we find

$$\left[ \begin{array}{cccc|c} 1 & 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 2 & 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array} \right] \mapsto \left[ \begin{array}{cccc|c} 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

The general solution is thus

$$\begin{bmatrix} r-s \\ -r \\ s \\ r \end{bmatrix} = r \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Because the eigenspace is 2-dimensional and the eigenvalue had multiplicity 2, the eigenvalue is not defective.

3. Suppose there are four teams in a curling league. At the end of the season, the results are as follows:

Team 1 beat teams 2 and 3, but lost to team 4.

Team 2 beat team 3, but lost to teams 1 and 4.

Team 3 beat team 4, but lost to teams 1 and 2.

Team 4 beat teams 1 and 2, but lost to team 3.

(a) Form the corresponding matrix  $A$  that reflects these results, where

$$a_{ij} = \begin{cases} 1 & \text{if team } i \text{ beat team } j \\ 0 & \text{otherwise} \end{cases}$$

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

(b) How small can the dominant eigenvalue for  $A$  be? How large? Explain.

The dominant eigenvalue is bounded above and below by the largest and smallest row sums of  $A$ . Thus we have  $1 \leq \lambda \leq 2$ .

(c) It turns out that the dominant eigenvalue is approximately 1.395, and the corre-

sponding eigenvector is  $\mathbf{v} = \begin{bmatrix} 0.552 \\ 0.321 \\ 0.448 \\ 0.626 \end{bmatrix}$ . How should the teams be ranked?

The teams should be ranked in order according to the relative size of the corresponding entry in the eigenvector associated to the dominant eigenvalue. Thus Team 4 is #1, Team 1 is #2, Team 3 is #3, and Team 2 is #4.

4. Is the matrix  $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$  diagonalizable? Explain.

No. Because the matrix is upper-triangular, we may read the eigenvalues off the diagonal. Thus  $\lambda = 0$  is an eigenvalue with multiplicity one, and  $\lambda = 1$  is an eigenvalue of multiplicity two. All the eigenvalues are real, so for diagonalizability we just need to make sure none of them is defective. The eigenvalue  $\lambda = 0$  cannot be defective because it has multiplicity one, so we need only

check that  $\lambda = 1$  has eigenspace of dimension two. To this end we note that

$$\left[ \begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

is already row-reduced, and a general solution looks like

$$\begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Thus the corresponding eigenspace is one-dimensional, so the eigenvalue is defective, so the matrix is not diagonalizable.

5. Find an invertible matrix  $P$  and a diagonal matrix  $D$  so that  $P^{-1}AP = D$ , where

$$A = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}.$$

For the eigenvalues, we have

$$(4 - \lambda)(3 - \lambda) - 6 = \lambda^2 - 7\lambda + 6 = (\lambda - 1)(\lambda - 6),$$

so the eigenvalues are  $\lambda = 1$  and  $\lambda = 6$ . For the eigenvectors we have

$$\left[ \begin{array}{cc|c} 3 & 2 & 0 \\ 3 & 2 & 0 \end{array} \right] \mapsto \left[ \begin{array}{cc|c} 3 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right],$$

which has general solution  $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ , and

$$\left[ \begin{array}{cc|c} -2 & 2 & 0 \\ 3 & -3 & 0 \end{array} \right] \mapsto \left[ \begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right],$$

which has general solution  $\begin{bmatrix} r \\ r \end{bmatrix} = r \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Thus we have

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}, \quad P = \begin{bmatrix} 2 & 1 \\ -3 & 1 \end{bmatrix}.$$