Week 7 worksheet solutions

1. Consider the vector space $\mathbf{R}^2$ endowed with the inner product given by

$$\left\langle \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right\rangle = ac - ad - bc + 5bd.$$

(a) Find the lengths of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$$\left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\| = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle^{1/2} = \sqrt{(1)(1) - (1)(0) - (0)(1) + 5(0)(0)} = 1$$
$$\left\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\| = \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle^{1/2} = \sqrt{(0)(0) - (0)(1) - (1)(0) + 5(1)(1)} = \sqrt{5}$$

(b) Find the cosine of the angle between $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$$\cos\theta = \frac{\left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle^{1/2} \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle^{1/2}} = \frac{-1}{\sqrt{5}}$$

2. Consider the vector space $C[0,1]$ of functions continuous between zero and one, endowed with the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt$.

(a) Find the length of $f(t) = e^t$.

$$\|e^t\| = \langle e^t, e^t \rangle^{1/2} = \int_0^1 e^{2t}\,dt = \sqrt{\frac{1}{2}(e^{2t}|_{t=0}^{t=1})} = \sqrt{\frac{1}{2}(e^2 - 1)}$$

(b) Find all values for a and b so that $g(t) = at + b$ is orthogonal to $f(t) = t + 1$.

$at + b$ and $t + 1$ are orthogonal precisely if $\langle at + b, t + 1 \rangle = 0$. Thus we solve

$$0 = \int_0^1 (at + b)(t + 1)\,dt = \int_0^1 (at^2 + (a + b)t + b)\,dt = \frac{a}{3} + \frac{a + b}{2} + b.$$

Thus in order for $at+b$ to be orthogonal to $t+1$, we need that

$$\frac{5a}{6} + \frac{3b}{2} = 0 \text{ or } 5a + 9b = 0.$$

So any values of a and b where $b = -\frac{5}{9}a$ will work.

3. Let W be the subspace of $\mathbf{R}^4$ spanned by $\begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -6 \\ 1 \\ 5 \\ -8 \end{bmatrix}$. Find a basis for $W^\perp$.

We are looking for a vector in $\mathbf{R}^4$ that dots with each of the two given vectors to give zero. Thus we solve

$$2a + 0b - 1c + 3d = 0 \text{ and } -6a + 1b + 5c - 8d = 0.$$

This turns into the following augmented matrix, which row reduces as shown:

$$\left[\begin{array}{cccc|c} 2 & 0 & -1 & 3 & 0 \\ -6 & 1 & 5 & -8 & 0 \end{array}\right] \mapsto \left[\begin{array}{cccc|c} 2 & 0 & -1 & 3 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{array}\right].$$

A general solution to this system has the form

$$\begin{bmatrix} (s-3r)/2 \\ -2s-r \\ s \\ r \end{bmatrix} = r \begin{bmatrix} -3/2 \\ -1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1/2 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

Thus these two vectors span $W^\perp$.

4. In $C[0,1]$, find the projection of $t+1$ onto t^2.

$$\texttt{proj}_{t^2}(t+1) = \frac{\langle t+1, t^2 \rangle}{\langle t^2, t^2 \rangle} t^2 = \frac{\int_0^1 (t^3+t^2)\,dt}{\int_0^1 t^4\,dt} t^2 = \frac{7/12}{1/5} t^2 = \frac{7}{60} t^2.$$

5. Let W be the subspace of $\mathbf{R}^3$ spanned by $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(a) Find the distance between $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ and the nearest vector in W.

Let $\mathbf{v}$ denote the given vector, and let $\mathbf{w}_1$ and $\mathbf{w}_2$ denote the two given vectors in W. Note that $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal. The nearest vector in W to $\mathbf{v}$ is the projection of $\mathbf{v}$ onto W, which is given by

$$\texttt{proj}_W \mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \mathbf{w}_1 + \frac{\langle \mathbf{v}, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle} \mathbf{w}_2 = \frac{4}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{2}{1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

The distance between $\mathbf{v}$ and W is just the distance between $\mathbf{v}$ and this projection. This in turn is the length of the difference of the two, which is given by

$$\left\| \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\| = \left\langle \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\rangle^{1/2} = \sqrt{1+1} = \sqrt{2}.$$

(b) Write $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ as $\mathbf{w} + \mathbf{u}$, where $\mathbf{w}$ is in W and $\mathbf{u}$ is in $W^{\perp}$.

We've already found the projection of $\mathbf{v}$ onto W, so we set $\mathbf{w}$ to be this projection. Then $\mathbf{u}$ is just the difference between them (which is automatically in $W^{\perp}$). Thus we have

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$