# TFAE Handout

**Notation:**

TFAE means "the following are equivalent;" i.e., if one is true, all are true
$A$ is an $n \times m$ matrix
$S$ is the set of columns of $A$ (which is a set of $m$ vectors in $\mathbf{R}^n$)
$L : \mathbf{R}^m \to \mathbf{R}^n$ is the linear map $L(\mathbf{x}) = A\mathbf{x}$
$A'$ is the row-reduced version of $A$

**TFAE, and can only (but don't have to) happen when $m \leq n$:**
(When $m < n$, this means $A$ is tall, $L$ goes from small to big, and $S$ is too small to be a basis.)

$A'$ : Every column of $A'$ has an initial term
$S$ : $S$ is linearly independent
$A$ : $\begin{cases} A\mathbf{x} = \mathbf{0} \text{ has only the trivial solution } \mathbf{x} = \mathbf{0} \\ \text{rank}(A) = m \end{cases}$
$L$ : $\begin{cases} L \text{ is 1–1} \\ \ker(L) \text{ is trivial} \\ \dim \ker(L) = 0 \end{cases}$

**TFAE, and can only (but don't have to) happen when $m \geq n$:**
(When $m > n$, this means $A$ is wide, $L$ goes from big to small, and $S$ is too big to be a basis.)

$A'$ : Every row of $A'$ is non-zero
$S$ : $S$ spans $\mathbf{R}^n$
$A$ : $\begin{cases} A\mathbf{x} = \mathbf{b} \text{ always has a solution} \\ \text{rank}(A) = n \end{cases}$
$L$ : $\begin{cases} L \text{ is onto} \\ \text{range}(L) = W \\ \dim \text{range}(L) = n \end{cases}$

**TFAE, and can only (but don't have to) happen when $m = n$:**

All of the above
$S$ is a basis for $\mathbf{R}^n$
$A$ row-reduces to the identity, and so is invertible/nonsingular
$L$ is an isomorphism, and so is invertible