Linear Algebra I   Test 1 Solutions   June 15, 2007

1. (a) Suppose W is a subspace of $\mathbf{R}^5$. What can you say (if anything) about the dimension of W?

> We know that $\dim(W) \leq 5$. This is because a basis for W consists of vectors that are linearly independent in $\mathbf{R}^5$, and any set with more than 5 vectors in $\mathbf{R}^5$ must be linearly dependent (this is a theorem).

(b) Suppose further that W consists of all vectors of the form $\begin{bmatrix} a+b+c \\ b-d \\ a+2b+2c+2d \\ -a+4d \\ 3b+c-d \end{bmatrix}$.

Find a set of vectors spanning W.

> The subspace W consists of all vectors of the form

$$\begin{bmatrix} a+b+c \\ b-d \\ a+2b+2c+2d \\ -a+4d \\ 3b+c-d \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} + d\begin{bmatrix} 0 \\ -1 \\ 2 \\ 4 \\ -1 \end{bmatrix}.$$

> It follows that these four vectors span W.

(c) Show using the definition that the set of all vectors of the form $\begin{bmatrix} a \\ b \\ a-b \\ 0 \end{bmatrix}$ is a subspace of $\mathbf{R}^4$.

> For closure under addition, we have

$$\begin{bmatrix} a_1 \\ b_1 \\ a_1-b_1 \\ 0 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ a_2-b_2 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1+a_2 \\ b_1+b_2 \\ a_1-b_1+a_2-b_2 \\ 0+0 \end{bmatrix} = \begin{bmatrix} a_1+a_2 \\ b_1+b_2 \\ (a_1+a_2)-(b_1+b_2) \\ 0 \end{bmatrix}.$$

> For closure under scalar multiplication, we have

$$\alpha\begin{bmatrix} a \\ b \\ a-b \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha a \\ \alpha b \\ \alpha(a-b) \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha a \\ \alpha b \\ \alpha a-\alpha b \\ 0 \end{bmatrix}.$$

2. Consider the linear map $L : \mathbf{R}^4 \to \mathbf{R}^4$ given by $L\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a+2b+c+d \\ b+d \\ 2a+3b+2c+d \\ a+c-d \end{bmatrix}$.

Find the standard matrix representation for L. Then verify the rank-nullity theorem for this map by finding bases for the image and for the kernel.

The standard matrix representation is

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 2 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It follows that the kernel consists of vectors of the form

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} r-s \\ -r \\ s \\ r \end{bmatrix} = r \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Thus these two vectors form a basis for the kernel. In particular, the kernel has dimension 2.

The range is spanned by the columns of A. Thus a basis for the range is formed by the vectors corresponding to columns that contain initial terms when row-reduced. So a basis for the range is

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \end{bmatrix}.$$

In particular, the range has dimension 2. $2+2=4$, which is the dimension of the domain $\mathbf{R}^4$, so that the rank-nullity theorem is verified.

3. (a) Suppose $L : V \to V$ is a linear map from a finite-dimensional vector space to itself. To say that L is *onto* means that the range of L is all of V. Use the rank-nullity theorem to show that if L is onto, then the kernel of L is trivial.

The rank-nullity theorem says the following:

$$\texttt{dim ker}(L) + \texttt{dim range}(L) = \texttt{dim}(V).$$

Because L is onto, the range is all of V, so its dimension is the same as V. Thus we have

$$\texttt{dim ker}(L) + \texttt{dim}(V) = \texttt{dim}(V),$$

which implies the kernel has dimension zero, and hence is trivial.

(b) To say a map is *one-to-one* means that different inputs have to produce different outputs. Show that if a linear map is not one-to-one, then it has nontrivial kernel. (**Hint:** The fact that L is not one-to-one means that there are two vectors $\mathbf{v} \neq \mathbf{u}$ so that $L(\mathbf{v}) = L(\mathbf{u})$. What is $L(\mathbf{v}-\mathbf{u})$? Why does this imply the kernel is nontrivial?)

As mentioned in the hint, because L is not one-to-one, there are vectors $\mathbf{v}$ and $\mathbf{u}$ so that $L(\mathbf{v}) = L(\mathbf{u})$, while $\mathbf{v} \neq \mathbf{u}$. But then because L is linear, we have

$$L(\mathbf{v}-\mathbf{u}) = L(\mathbf{v}) - L(\mathbf{u}) = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Also, because $\mathbf{v} \neq \mathbf{u}$, we have $\mathbf{v}-\mathbf{u} \neq \mathbf{0}$. Thus we have found a non-zero vector that is mapped to the zero vector by L. In other words, the kernel is nontrivial.

4. True/False. If true, explain. If false, give a counterexample.

(a) Any set of vectors spanning $\mathbf{R}^n$ must contain at least n vectors.

True. Every spanning set contains a basis, and any basis contains n vectors. Thus the spanning set must have contained at least n vectors to begin with.

(b) Any set with more than n vectors in $\mathbf{R}^n$ must span $\mathbf{R}^n$.

False. A set consisting of several multiples of the same vector will not span $\mathbf{R}^n$ (as long as $n \neq 1$).