Linear Algebra I Test 2 Solutions June 29, 2007

1. Suppose $L : \mathbf{R}^3 \to \mathbf{R}^2$ is a linear map, $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbf{R}^3$, $T = \left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbf{R}^2$, and $A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 0 & 1 \end{bmatrix}$ is the matrix representation for L with respect to S and T. Find $L\left(\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \right)$.

Let $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$. We cannot apply L directly, because we do not have a formula for it. What we do have is a matrix A that ''does L,'' but only with S-coordinates as inputs and T-coordinates as outputs. Thus we first find the S-coordinates of $\mathbf{v}$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 4 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \Rightarrow \quad [\mathbf{v}]_S = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

Now we can ''do L'' by multiplying by A:

$$[L(\mathbf{v})]_T = A[\mathbf{v}]_S = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

Because A outputs T-coordinates, these are these T-coordinates of $L(\mathbf{v})$. Thus to find $L(\mathbf{v})$ itself, we use the fact that ''coordinates = coefficients'' to obtain:

$$L(\mathbf{v}) = 5 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ 21 \end{bmatrix}.$$

2. Consider the linear map $L : \mathbf{R}^3 \to \mathbf{R}^3$ given by $L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} a + 2b + c \\ 2a + 3b + 2c \\ a + c \end{bmatrix}$. Find the matrix representation for L with respect to the basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

If we let A denote the matrix representation of L with respect to the given basis (call it S), then the columns of A are the S-coordinates of L of the S-basis vectors. Thus we have

$$\left[L\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) \right]_S = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}_S = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix},$$

$$\left[L\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)\right]_S = \begin{bmatrix}2\\3\\0\end{bmatrix}_S = \begin{bmatrix}0\\1\\2\end{bmatrix},$$

$$\left[L\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)\right]_S = \begin{bmatrix}3\\5\\1\end{bmatrix}_S = \begin{bmatrix}1\\3\\2\end{bmatrix},$$

where the S-coordinates giving the rightmost equalities were found via the following row-reduction:

$$\begin{bmatrix}1 & 0 & 1 & | & 2 & 2 & 3\\0 & 1 & 1 & | & 4 & 3 & 5\\1 & 0 & 0 & | & 2 & 0 & 1\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & 0 & 0 & | & 2 & 0 & 1\\0 & 1 & 0 & | & 4 & 1 & 3\\0 & 0 & 1 & | & 0 & 2 & 2\end{bmatrix}.$$

It follows that

$$A = \begin{bmatrix}2 & 0 & 1\\4 & 1 & 3\\0 & 2 & 2\end{bmatrix}.$$

3. Suppose $L : \mathbf{R}^3 \to \mathbf{R}^2$ is a linear map whose matrix representation with respect to S and T is $\begin{bmatrix}1 & 1 & 0\\2 & 1 & 1\end{bmatrix}$. Suppose we have transition matrices $P_{S' \leftarrow S} = \begin{bmatrix}1 & 2 & 0\\1 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ and $Q_{T' \leftarrow T} = \begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}$. Find the matrix representation for L with respect to S' and T'.

The matrix representation B for L with respect to S' and T', thought of as a function, has S'-coordinates as inputs and T'-coordinates as outputs. Thus to ''do B'' using the given information, we need to take S'-coordinates as inputs, turn them into S-coordinates (using P^{-1}), apply L (multiply by the given matrix A), then turn the T-coordinate outputs of A into T'-coordinates (using Q). In other words,

$$B = QAP^{-1} = \begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\2 & 1 & 1\end{bmatrix}\begin{bmatrix}-1 & 2 & 0\\1 & -1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}-3 & 11 & 3\\-5 & 19 & 5\end{bmatrix},$$

where P^{-1} is found via the following row-reduction:

$$\begin{bmatrix}1 & 2 & 0 & | & 1 & 0 & 0\\1 & 1 & 0 & | & 0 & 1 & 0\\0 & 0 & 1 & | & 0 & 0 & 1\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & 0 & 0 & | & -1 & 2 & 0\\0 & 1 & 0 & | & 1 & -1 & 0\\0 & 0 & 1 & | & 0 & 0 & 1\end{bmatrix}.$$