Linear Algebra I **Worksheet 1 Solutions** **June 8, 2007**

1. [3 pts] Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 2 \\ 1 & 0 \\ 3 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 1 & 2 \\ -4 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix} \qquad E = \begin{bmatrix} 2 & 0 & 3 \\ -4 & 1 & 2 \\ 5 & 4 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} -1 & 2 \\ 0 & 4 \\ 3 & 5 \end{bmatrix}$$

If possible, compute: (a) $DA + B$ (b) EC (c) CE (d) $EB + F$

`(a) Not possible; (b)` $\begin{bmatrix} 13 & 8 & 7 \\ -6 & 0 & -5 \\ -3 & 7 & 15 \end{bmatrix}$;

`(c)` $\begin{bmatrix} 10 & 9 & 10 \\ -3 & 4 & -11 \\ 3 & 6 & 14 \end{bmatrix}$; `(d)` $\begin{bmatrix} 12 & 0 \\ -1 & -8 \\ 20 & 13 \end{bmatrix}$

2. [12 pts] Let $L : \mathbf{R}^4 \to \mathbf{R}^3$ be the linear transformation defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x - z \\ y \\ w + z \end{bmatrix}.$$

(a) Show directly that L is linear.

(1) $$L\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \\ w_1 + w_2 \end{bmatrix}\right) = \begin{bmatrix} (x_1 + x_2) - (z_1 + z_2) \\ y_1 + y_2 \\ (w_1 + w_2) + (z_1 + z_2) \end{bmatrix}$$

$$= \begin{bmatrix} (x_1 - z_1) + (x_2 - z_2) \\ y_1 + y_2 \\ (w_1 + z_1) + (w_2 + z_2) \end{bmatrix} = \begin{bmatrix} x_1 - z_1 \\ y_1 \\ w_1 + z_1 \end{bmatrix} + \begin{bmatrix} x_2 - z_2 \\ y_2 \\ w_2 + z_2 \end{bmatrix} = L\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{bmatrix}\right).$$

(2) $$L\left(\alpha \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = L\left(\begin{bmatrix} \alpha x \\ \alpha y \\ \alpha z \\ \alpha w \end{bmatrix}\right) = \begin{bmatrix} \alpha x - \alpha z \\ \alpha y \\ \alpha w + \alpha z \end{bmatrix} = \alpha \begin{bmatrix} x - z \\ y \\ w + z \end{bmatrix} = \alpha L\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right).$$

(b) Find the standard matrix representation for L.

We plug in the standard basis vectors and line up the results:

$$L(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad L(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$L(\mathbf{e}_3) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad L(\mathbf{e}_4) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus the matrix is $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

(c) Describe the kernel of L.

We set the output equal to the zero vector and solve. This gives the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right],$$

the solution to which is $w = r$, $z = -w = -r$, $y = 0$, and $x = z = -r$, so that a typical element of the kernel is $\begin{bmatrix} -r \\ 0 \\ -r \\ r \end{bmatrix}$.

3. [4 pts] Find an equation relating a, b, and c, so that the following linear system is consistent for any values of a, b, and c that satisfy that equation:

$$\begin{aligned} x \quad\quad + z &= a \\ x + y + 2z &= b \\ 2x + y + 3z &= c \end{aligned}$$

We solve the system by row-reducing, as follows (for example):

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 1 & 1 & 2 & b \\ 2 & 1 & 3 & c \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 1 & b-a \\ 0 & 1 & 1 & c-2a \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 0 & c-2a-(b-a) \end{array}\right].$$

The only way this system has a solution is if $c-2a-(b-a)$ is zero. In particular, we must have $c = a + b$, for example.

4. [2 pts] Is it possible for a system of three equations in two unknowns to be consistent? If so, give an explicit example; if not, explain why not.

`Yes. For example,` $\begin{cases} x + y = 0 \\ 2x + 2y = 0 \\ 3x + 3x = 0 \end{cases}$.