Linear Algebra I **Worksheet 2 Solutions** **June 13, 2007**

1. (a) [3 pts] Show that the set of all vectors of the form $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, where $c = a + 2b$ and $d = a - 3b$, is a subspace of $\mathbf{R}^4$.

`We went over the direct way of doing this today in class. Here's a shortcut way:`
`This set is the set of all vectors of the form`

$$\begin{bmatrix} a \\ b \\ a+2b \\ a-3c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}.$$

`But this is precisely the span of the two rightmost vectors in the equation above. We have a theorem that says that spans are always subspaces.`

(b) [3 pts] Show that the set of all solutions to the equation $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} \neq \mathbf{0}$, is not a subspace of $\mathbf{R}^n$.

Suppose $\mathbf{x}_1$ and $\mathbf{x}_2$ are both solutions to this equation. Then

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{b} + \mathbf{b} = 2\mathbf{b} \neq \mathbf{b},$$

where the last inequality is because we know that $\mathbf{b} \neq \mathbf{0}$. Thus the sum of two solutions is not again a solution, so the solution set is not closed under addition, hence not a subspace.

2. Consider the linear map $L(\mathbf{v}) = A\mathbf{v}$, where $A = \begin{bmatrix} 1 & 3 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 2 & 5 & 3 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$.

`Before anything else, let's row reduce this matrix (we know we're going to have to eventually):`

$$\begin{bmatrix} 1 & 3 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 2 & 5 & 3 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) [6 pts] Find a linearly independent set of vectors that span the range of L.

The range is all the outputs. A typical output looks like

$$\begin{bmatrix} 1 & 3 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 2 & 5 & 3 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \\ u \end{bmatrix} = \begin{bmatrix} x+3y+2z+w+u \\ x+2y+z+w \\ 2x+5y+3z+w+u \\ y+z+u \end{bmatrix}$$

$$= x\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + y\begin{bmatrix} 3 \\ 2 \\ 5 \\ 1 \end{bmatrix} + z\begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} + w\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Thus the range is spanned by these five vectors. PLEASE NOTE: These five vectors are precisely the columns of A. THIS IS ALWAYS TRUE: the columns of A span the range of the map $L(\mathbf{v}) = A\mathbf{v}$.
Now we just need to find a maximal independent subset of these.
For this we line them all up and row reduce. The vectors corresponding to columns with initial terms form the set we're looking for.
But because these vectors are the columns of A, lining them up and row reducing is the same as row reducing A, which we did above already. The initial terms end up in columns 1, 2, and 4. Thus vectors 1, 2, and 4 form a basis for the range. I.e., a basis for the range of L is given by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(b) [6 pts] Find a basis for $\mathbf{R}^4$ that contains your vectors from part (a).

To extend this set of three vectors to a basis for all of $\mathbf{R}^4$, we tack on the standard basis for $\mathbf{R}^4$ to produce a (too large) spanning set, and then row-reduce, looking for initial terms. This gives the following:

$$\begin{bmatrix} 1 & 3 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ 2 & 5 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 \end{bmatrix}.$$

Because the initial terms are in columns 1, 2, 3, and 4, these vectors form a basis for $\mathbf{R}^4$. So our basis is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

3. [3 pts] For what values of c are the vectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ c^2+2 \end{bmatrix}$ linearly dependent?

The most general way is as follows: By the definition of linear dependence, we are asking when the following system has a nontrivial solution:

$$a \begin{bmatrix} 1 \\ 3 \end{bmatrix} + b \begin{bmatrix} 2 \\ c^2+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This means row-reducing as follows:

$$\begin{bmatrix} 1 & 2 & \mid & 0 \\ 3 & c^2+2 & \mid & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & \mid & 0 \\ 0 & c^2-4 & \mid & 0 \end{bmatrix}.$$

This system has nontrivial solutions precisely when $c^2-4=0$, or when $c=\pm 2$.

A simpler way of doing this (that depends on the fact that there were only two vectors involved) is to note that for pairs of vectors, linearly dependent is equivalent to parallel, which is equivalent to being multiples of one another. So these are dependent precisely when they are multiples of one another. If they are going to be multiples, it must be that the second is twice the first (from looking at the top entry), thus we have that $c^2+2=6$, and we get the same solutions as before.