Linear Algebra I **Worksheet 3 Solutions** **June 22, 2007**

1. Consider the linear map $L : \mathbf{R}^4 \to \mathbf{R}^4$ given by $L\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} b-d \\ a+b+c \\ -a+4d \\ 3b+c-d \end{bmatrix}$.

(a) [3 pts] Find the standard matrix representation A for L.

$$A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 4 \\ 0 & 3 & 1 & -1 \end{bmatrix}$$

(b) [4 pts] Show that L is an isomorphism by finding the inverse of the matrix A from part (a).

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 4 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 8/3 & 4/3 & 1/3 & -4/3 \\ 0 & 1 & 0 & 0 & 5/3 & 1/3 & 1/3 & -1/3 \\ 0 & 0 & 1 & 0 & -13/3 & -2/3 & -2/3 & 5/3 \\ 0 & 0 & 0 & 1 & 2/3 & 1/3 & 1/3 & -1/3 \end{array}\right]$$

So

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 8 & 4 & 1 & -4 \\ 5 & 1 & 1 & -1 \\ -13 & -2 & -2 & 5 \\ 2 & 1 & 1 & -1 \end{bmatrix}.$$

(c) [3 pts] Use your answer to part (b) to write a formula for $L^{-1} : \mathbf{R}^4 \to \mathbf{R}^4$ like that for L above.

$$L^{-1}\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} 8a+4b+c-4d \\ 5a+b+c-d \\ -13a-2b-2c+5d \\ 2a+b+c-d \end{bmatrix}.$$

2. Consider the basis $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ for $\mathbf{R}^3$.

(a) [3 pts] Suppose $[\mathbf{v}]_S = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $\mathbf{v}$.

$$\mathbf{v} = 1\begin{bmatrix}1\\1\\1\end{bmatrix} + 2\begin{bmatrix}1\\0\\1\end{bmatrix} + 3\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}3\\4\\6\end{bmatrix}.$$

(b) [3 pts] Find the S-coordinates for $\begin{bmatrix}1\\2\\3\end{bmatrix}$.

$$\left[\begin{array}{ccc|c}1 & 1 & 0 & 1\\1 & 0 & 1 & 2\\1 & 1 & 1 & 3\end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\1 & 0 & 1 & 1\\1 & 1 & 1 & 2\end{array}\right],$$

so

$$\begin{bmatrix}1\\2\\3\end{bmatrix}_S = \begin{bmatrix}0\\1\\2\end{bmatrix}.$$

3. [5 pts] Show that $[\mathbf{v}]_S + [\mathbf{w}]_S = [\mathbf{v}+\mathbf{w}]_S$ for any two vectors $\mathbf{v}$ and $\mathbf{w}$ and any basis S.

Suppose $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, and set $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$ and $\mathbf{w} = b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n$, so that $[\mathbf{v}]_S = \begin{bmatrix}a_1\\\vdots\\a_n\end{bmatrix}$ and $[\mathbf{w}]_S = \begin{bmatrix}b_1\\\vdots\\b_n\end{bmatrix}$. Then the left side of the equation becomes

$$[\mathbf{v}]_S + [\mathbf{w}]_S = \begin{bmatrix}a_1\\\vdots\\a_n\end{bmatrix} + \begin{bmatrix}b_1\\\vdots\\b_n\end{bmatrix} = \begin{bmatrix}a_1+b_1\\\vdots\\a_n+b_n\end{bmatrix}.$$

On the other hand, note that

$\mathbf{v}+\mathbf{w} = (a_1\mathbf{v}_1+\cdots+a_n\mathbf{v}_n)+(b_1\mathbf{v}_1+\cdots+b_n\mathbf{v}_n) = (a_1+b_1)\mathbf{v}_1+\cdots+(a_n+b_n)\mathbf{v}_n,$

so that the right side of the equation also is

$$[\mathbf{v}+\mathbf{w}]_S = \begin{bmatrix}a_1+b_1\\\vdots\\a_n+b_n\end{bmatrix}.$$