Linear Algebra I **Worksheet 4 Solutions** **June 27, 2007**

1. Consider the following bases for $\mathbf{R}^3$:

$$T = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \qquad T' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(a) Find the transition matrix $Q_{T \leftarrow T'}$.

To get the columns of Q, we need to find the T-coordinates of the T'-basis vectors. We can simultaneously find all three sets of coordinates by row-reducing as follows:

$$\begin{bmatrix} -1 & 0 & 1 & | & 1 & 0 & 0 \\ 1 & 1 & 0 & | & 1 & 1 & 0 \\ 0 & 1 & 0 & | & 1 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & | & 0 & 0 & -1 \\ 0 & 1 & 0 & | & 1 & 1 & 1 \\ 0 & 0 & 1 & | & 1 & 0 & -1 \end{bmatrix}.$$

Thus we have

$$Q_{T \leftarrow T'} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

(b) Find the inverse of Q.

We invert as usual, obtaining

$$\begin{bmatrix} 0 & 0 & -1 & | & 1 & 0 & 0 \\ 1 & 1 & 1 & | & 0 & 1 & 0 \\ 1 & 0 & -1 & | & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & | & -1 & 0 & 1 \\ 0 & 1 & 0 & | & 2 & 1 & -1 \\ 0 & 0 & 1 & | & -1 & 0 & 0 \end{bmatrix},$$

so that

$$Q^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 0 \end{bmatrix}.$$

2. Given the basis $S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ for $\mathbf{R}^2$, suppose $S' = \{\mathbf{v}_1, \mathbf{v}_2\}$ is another basis for $\mathbf{R}^2$, so that the transition matrix $P_{S \leftarrow S'}$ is $\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$.

(a) Find the basis S'.

The columns of P are the S-coordinates of the S'-basis vectors. Thus we have

$$[\mathbf{v}_1]_S = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_1 = \frac{2}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and

$$[\mathbf{v}]_S = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_2 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(b) Suppose $[\mathbf{v}]_{S'} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Use matrix multiplication to find $[\mathbf{v}]_S$.

Because P is the transition matrix from the S'-basis to the S-basis, we know that for all vectors $\mathbf{v}$, we have $[\mathbf{v}]_S = P[\mathbf{v}]_{S'}$. In particular, this gives

$$[\mathbf{v}]_S = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

3. Let $L : \mathbf{R}^2 \to \mathbf{R}^3$ be defined by $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} -a \\ a-b \\ a \end{bmatrix}$.

(a) Find the matrix representation for L with respect to S and T, given above.

The columns of the required matrix are the T-coordinate vectors of L of the S-basis vectors. Thus we have that the columns are

$$\left[L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)\right]_T = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}_T = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix},$$

$$\left[L\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)\right]_T = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}_T = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

so that the matrix representation is

$$A = \begin{bmatrix} -2 & 1 \\ 1 & 1 \\ -3 & 0 \end{bmatrix}.$$

Note that the equalities above were found by directly plugging in to L, while the second equalities (the T-coordinates) were found by simultaneously solving via the following row-reduction:

$$\left[\begin{array}{ccc|cc} -1 & 0 & 1 & -1 & -1 \\ 1 & 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|cc} 1 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -3 & 0 \end{array}\right].$$

(b) Use matrix multiplication to find $[L(\mathbf{v})]_T$, where $\mathbf{v}$ is as above.

By definition of the matrix A above, we have

$$[L(\mathbf{v})]_T = A[\mathbf{v}]_S = \begin{bmatrix} -2 & 1 \\ 1 & 1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ -6 \end{bmatrix}.$$

(c) Use matrix multiplication to find $[L(\mathbf{v})]_{T'}$.

Note that because Q from problem #1 is the transition matrix from the T'-basis to the T-basis, it follows that Q^{-1} is the transition from the T-basis to the T'-basis, so we have

$$[L(\mathbf{v})]_{T'} = Q^{-1}[L(\mathbf{v})]_T = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ 3 \\ -6 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix}.$$