**Linear Algebra I  Worksheet 5 Solutions  July 6, 2007**

1. Suppose $V$ is the set of all positive real numbers, and define operations $\oplus$ and $\odot$ as follows:

- if $\mathbf{u}$ is the positive real number $a$, and $\mathbf{v}$ is the positive real number $b$, then define $\mathbf{u} \oplus \mathbf{v}$ to be $ab$ (so $\oplus$ is actually multiplication)
- if $\mathbf{u}$ is the positive real number $a$, then define $\alpha \odot \mathbf{u}$ to be $a^\alpha$ (so $\odot$ is actually exponentiation)

Show that $V$ with these operations is a vector space, by showing that it is closed under $\oplus$ and $\odot$, and showing that it satisfies properties 1–8 in the definition of a real vector space.

Throughout, we will assume that $\mathbf{u}$ represents the positive real number $a$, $\mathbf{v}$ is the positive real number $b$, and $\mathbf{w}$ is the positive real number $c$.

(a) $\mathbf{u} \oplus \mathbf{v}$ is $ab$, which is again a positive real number.

(1) $\mathbf{u} \oplus \mathbf{v}$ is $ab$, while $\mathbf{v} \oplus \mathbf{u}$ is $ba$, and $ab = ba$.

(2) $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w})$ is $a(bc)$, while $(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$ is $(ab)c$, and $a(bc) = (ab)c$.

(3) Let $\mathbf{0}$ be the positive real number $1$. Then $\mathbf{u} \oplus \mathbf{0} = \mathbf{0} \oplus \mathbf{u} = \mathbf{u}$ because $a1 = 1a = a$.

(4) Let $-\mathbf{u}$ be the positive real number $\frac{1}{a}$. Then $\mathbf{u} \oplus -\mathbf{u} = a\frac{1}{a} = 1 = \mathbf{0}$.

(b) $\alpha \odot \mathbf{u} = a^\alpha$, which is again a positive real number for any real number $\alpha$.

(5) $\alpha \odot (\mathbf{u} \oplus \mathbf{v}) = (ab)^\alpha = a^\alpha b^\alpha = \alpha \odot \mathbf{u} \oplus \alpha \odot \mathbf{v}$.

(6) $(\alpha + \beta) \odot \mathbf{u} = a^{\alpha+\beta} = a^\alpha a^\beta = \alpha \odot \mathbf{u} \oplus \beta \odot \mathbf{u}$

(7) $\alpha \odot (\beta \odot \mathbf{u}) = (a^\alpha)^\beta = a^{\alpha\beta} = (\alpha\beta) \odot \mathbf{u}$.

(8) $1 \odot \mathbf{u} = a^1 = a = \mathbf{u}$.

2. (a) Show that the set of all continuous functions $f(x)$ with the property that $f(3) = 0$ is a subspace of $C(-\infty, \infty)$.

We need to show that this set is closed under addition and scalar multiplication. To this end, suppose $f$ and $g$ are two functions in $C(-\infty, \infty)$ with the property that $f(3) = 0$ and $g(3) = 0$. We need to show that $f+g$ also has this property, namely, that $(f+g)(3) = 0$. But $(f+g)(3) = f(3) + g(3) = 0 + 0 = 0$. We also need that $\alpha f$ has this property, but $(\alpha f)(3) = \alpha f(3) = \alpha 0 = 0$. Thus this set is a subspace.

(b) Show that the set of all continuous functions $f(x)$ with the property that $f(0) = 3$ is not a subspace of $C(-\infty, \infty)$.

Suppose $f(0) = 3$ and $g(0) = 3$. Then $(f+g)(0) = f(0) + g(0) = 3 + 3 = 6$, so this set is not closed under addition.

3. (a) Do the matrices $\left\{ \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -5 & 2 \end{bmatrix} \right\}$ span $M_{22}$?

We'll use the standard basis for $M_{22}$ to turn these matrices into columns, and answer the question in $\mathbf{R}^4$ instead. So the new question is, do the following column vectors span $\mathbf{R}^4$:

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -5 \\ 2 \end{bmatrix}.$$

For this, we line up and row reduce:

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 2 & -1 & 0 & -5 \\ 1 & 2 & 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It follows that these four columns span $\mathbf{R}^4$ (because there are no zero rows), so the original four matrices span $M_{22}$.

(b) Are the polynomials $\{t^3 + t^2, t + 1, t^3 + 1, t^2 + t + 1\}$ linearly independent in $P_3$?

We use standard coordinates to answer the question about columns instead. The new question is: are the following column vectors linearly independent in $\mathbf{R}^4$:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

For this, we line them up and row-reduce:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It follows that these four columns are linearly independent (because every column has a leading term), and so the original four polynomials are linearly independent in $P_3$.

4. Let $W$ be the subspace of $P_2$ spanned by

$$S = \{t^2 + 2t + 2,\ 3t^2 + 2t + 1,\ 11t^2 + 10t + 7,\ 7t^2 + 6t + 4\}.$$

(a) Find the coordinates of these vectors with respect to the basis $\{t^2+t+1, t^2+t, t^2\}$.

We use standard coordinates to turn this into the following question: Let $W$ be the subspace of $\mathbf{R}^3$ spanned by

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 11 \\ 10 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix} \right\}.$$

Find the coordinates of these vectors with respect to the basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

For this, we row-reduce as follows:

$$\left[\begin{array}{ccc|cccc} 1 & 1 & 1 & 1 & 3 & 11 & 7 \\ 1 & 1 & 0 & 2 & 2 & 10 & 6 \\ 1 & 0 & 0 & 2 & 1 & 7 & 4 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 2 & 1 & 7 & 4 \\ 0 & 1 & 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 & 1 \end{array}\right].$$

So if we call the basis $T$, the coordinates are:

$$[t^2 + 2t + 2]_T = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \qquad [3t^2 + 2t + 1]_T = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$[11t^2 + 10t + 7]_T = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}, \qquad [7t^2 + 6t + 4]_T = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}.$$

(b) Use the coordinate vectors found in part (a) to find a basis for $W$ consisting of vectors from $S$.

We line them up and row-reduce as follows:

$$\begin{bmatrix} 2 & 1 & 7 & 4 \\ 0 & 1 & 3 & 2 \\ -1 & 1 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 1 & 7 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It follows that the first two columns form a basis for the subspace spanned by all the columns, so the first two polynomials form a basis for the subspace spanned by all the polynomials. In other words, the basis for $W$ we seek is

$$\{t^2 + 2t + 2,\ 3t^2 + 2t + 1\}.$$