Linear Algebra I **Worksheet 6 Solutions** **July 12, 2007**

1. Find the determinants of the following matrices:

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 3 & 2 & 5 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 3 & 5 & 2 & 9 & 0 \\ 5 & 3 & 9 & 5 & 1 \\ 0 & 1 & 4 & 2 & 9 \\ 1 & 8 & 5 & 2 & 0 \\ 5 & 3 & 9 & 5 & 1 \end{bmatrix}$$

(a) -10; (b) -5; (c) 24 (the product of the diagonal entries, because it's upper triangular); (d) 0 (rows 2 and 5 are equal).

2. The map $L\colon P_2 \to P_2$ given by $L(p(t)) = 2p'(t) + t^2 p(0)$ has only one eigenvalue $\lambda = 2$. Find the dimension of the corresponding eigenspace. Do this directly, without using matrices.

If we write $p(t) = at^2 + bt + c$, then the map L is given by

$$L(at^2 + bt + c) = 2(2at + b) + t^2(c) = ct^2 + 4at + 2b.$$

The eigenvectors for L corresponding to the eigenvalue $\lambda = 2$ are precisely those polynomials $p(t)$ so that $L(p(t)) = 2p(t)$. In other words $ct^2 + 4at + 2b = 2at^2 + 2bt + 2c$. Setting corresponding coefficients equal, we find that $c = b = 2a$. So the general eigenvector looks like $rt^2 + 2rt + 2r$. This is a 1-dimensional space.

3. (a) Find eigenvalues and eigenvectors of the following matrix:

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & -2 \\ 2 & 0 & 5 \end{bmatrix}$$

For the eigenvalues, we have

$$\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & 0 & 0 \\ -2 & 3-\lambda & -2 \\ 2 & 0 & 5-\lambda \end{bmatrix} = (3-\lambda)^2(5-\lambda).$$

Thus the eigenvalues are $\lambda = 3$ (twice) and $\lambda = 5$.

When $\lambda = 3$, eigenvectors are solutions to the following:

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -2 & 0 & -2 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so a general solution is

$$\begin{bmatrix} -r \\ s \\ r \end{bmatrix} = r \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

These last two vectors form a basis for the eigenspace.

When $\lambda = 5$, eigenvectors are solution to the following:

$$\begin{bmatrix} -2 & 0 & 0 & | & 0 \\ -2 & -2 & -2 & | & 0 \\ 2 & 0 & 0 & | & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix},$$

so a general solution is

$$\begin{bmatrix} 0 \\ -r \\ r \end{bmatrix} = r \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

This vector is a basis for the eigenspace.

(b) Find a diagonal matrix D and an invertible matrix P so that $D = P^{-1}AP$.

All eigenvalues are real and none are defective, so A is diagonalizable.
One choice for D and P is

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \qquad P = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$