1. Compute e^A, where $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$.

First we find the eigenvalues of A:

$$\text{det}(A - \lambda I) = (\lambda - 2)(\lambda - 3),$$

so eigenvalues are $\lambda = 2$ and $\lambda = 3$. Next we solve the system:

$$\left[\begin{array}{cc|c} 3 & 1 & e^3 \\ 2 & 1 & e^2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & e^3 - e^2 \\ 0 & 1 & 3e^2 - 2e^3 \end{array}\right].$$

It follows that

$$e^A = (e^3 - e^2)A + (3e^2 - 2e^3)I = \begin{bmatrix} 2e^2 - e^3 & e^3 - e^2 \\ 2e^2 - 2e^3 & 2e^3 - e^2 \end{bmatrix}.$$

2. Suppose there are four teams in a league. At the end of the season, the results are as follows:

Team 1 beat teams 2 and 3, but lost to team 4.
Team 2 beat team 3, but lost to teams 1 and 2.
Team 3 beat team 4, but lost to teams 1 and 2.
Team 4 beat teams 1 and 2, but lost to team 3.

(a) Form the corresponding matrix A that reflects these results.

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

(b) How small can the dominant eigenvalue for A be? How large? Explain.

The dominant eigenvalue is between the smallest and largest row sums of the matrix, so it's no smaller than 1 and no bigger than 2.

(c) It turns out that the dominant eigenvalue is approximately 1.395, and the corresponding eigenvector is $\mathbf{v} = \begin{bmatrix} 0.552 \\ 0.321 \\ 0.448 \\ 0.626 \end{bmatrix}$. How should the teams be ranked?

First - team 4;
second - team 1;
third - team 3;
fourth - team 2.

3. Find a diagonal matrix D and an orthogonal matrix P so that $D = P^T AP$, where

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 3 \end{bmatrix}.$$

First we find eigenvalues:

$$\texttt{det}(A - \lambda I) = -(\lambda + 2)(\lambda - 4)(\lambda + 1),$$

so eigenvalues are $\lambda = -2$, $\lambda = -1$, and $\lambda = 4$, so

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Now we find eigenvectors. For $\lambda = -2$ we have

$$\left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 5 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so an eigenvector is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. For $\lambda = -1$ we have

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & -1 & 0 & 0 \\ -2 & 0 & 4 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so an eigenvector is $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. For $\lambda = 4$ we have

$$\left[\begin{array}{ccc|c} -4 & 0 & -2 & 0 \\ 0 & -6 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so an eigenvector is $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$. To make P orthogonal, we need to divide these by their lengths, so that we have

$$P = \begin{bmatrix} 0 & 2/\sqrt{5} & 1/\sqrt{5} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}.$$