Linear Algebra I Worksheet 8 Solutions Not Due

1. Consider the inner product space $\mathbf{R}^2$ with $\left\langle \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right\rangle = ac - ad - bc + 3bd.$

(a) Find the length of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

`Using the formula` $\| \mathbf{v} \| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$`, we have`

$$\| \begin{bmatrix} 2 \\ 3 \end{bmatrix} \| = \sqrt{4 - 6 - 6 + 27} = \sqrt{19}.$$

(b) Find the angle between $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

`Using the formula` $\theta = \cos^{-1}\left(\frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\| \mathbf{v} \| \| \mathbf{w} \|}\right)$`, we have`

$$\theta = \cos^{-1}\left(\frac{1 - 1 - 0 + 0}{(1)(\sqrt{2})}\right) = 90^\circ. \texttt{ (Weird, huh?)}$$

(c) Find the projection of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ onto $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

`Using the formula` $\texttt{proj}_{\mathbf{w}}\mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle}\mathbf{w}$`, we have`

$$\texttt{proj}_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{-1}{3}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/3 \end{bmatrix}.$$

2. Consider the inner product space P_2 with $\langle p(t), q(t) \rangle = \int_0^1 p(t)q(t)\,dt$.
(a) Let $W = \text{span}\{1, t\}$. Find an orthonormal basis for $W^\perp$.

`On second thought, this is probably easier to think about without the hint, so we'll ignore it for the time being. A polynomial` $p(t) = at^2 + bt + c$ `is in` $W^\perp$ `if it is orthogonal to both` 1 `and` t`. Thus we have that both of the following equations are satisfied:`

$$\langle p(t), 1 \rangle = \int_0^1 at^2 + bt + c\,dt = \frac{a}{3} + \frac{b}{2} + c = 0$$

$$\langle p(t), t \rangle = \int_0^1 at^3 + bt^2 + ct\,dt = \frac{a}{4} + \frac{b}{3} + \frac{c}{2} = 0.$$

`Because anything in` $W^\perp$ `has to satisfy both of these, we solve the corresponding system:`

$$\begin{bmatrix} 1/3 & 1/2 & 1 & | & 0 \\ 1/4 & 1/3 & 1/2 & | & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 3 & 6 & | & 0 \\ 0 & 1 & 6 & | & 0 \end{bmatrix}.$$

Thus the general solution is $c = r$, $b = -6r$, and $a = 6r$, so $W^\perp$ is spanned by $6t^2-6t+1$. We now have a basis for $W^\perp$. Normally, we would now need to do Gram-Schmidt to make it orthogonal, but because it is one-dimensional, we don't need to do that. We do, however, still need to scale to make the vector have unit length, so we calculate

$$\| 6t^2 - 6t + 1 \| = \sqrt{\int_0^1 36t^4 - 72t^3 + 48t^2 - 12t + 1\, dt} = \frac{1}{\sqrt{5}}$$

so our orthonormal basis is

$$6\sqrt{5}t^2 - 6\sqrt{5}t + \sqrt{5}.$$

(b) Find the matrix associated to the inner product.

Using the formulas (and the standard basis $\{t^2, t, 1\}$), we have

$$C = \begin{bmatrix} \langle t^2, t^2 \rangle & \langle t^2, t \rangle & \langle t^2, 1 \rangle \\ \langle t, t^2 \rangle & \langle t, t \rangle & \langle t, 1 \rangle \\ \langle 1, t^2 \rangle & \langle 1, t \rangle & \langle 1, 1 \rangle \end{bmatrix} = \begin{bmatrix} 1/5 & 1/4 & 1/3 \\ 1/4 & 1/3 & 1/2 \\ 1/3 & 1/2 & 1 \end{bmatrix}.$$

3. Find the degree 2 Fourier polynomial for $f(t) = \sin^2(t)$.

$$\begin{aligned} &\frac{\langle \sin^2(t), 1 \rangle}{\langle 1, 1 \rangle} 1 + \frac{\langle \sin^2(t), \cos(t) \rangle}{\langle \cos(t), \cos(t) \rangle} \cos(t) + \frac{\langle \sin^2(t), \sin(t) \rangle}{\langle \sin(t), \sin(t) \rangle} \sin(t) \\ &+ \frac{\langle \sin^2(t), \cos(2t) \rangle}{\langle \cos(2t), \cos(2t) \rangle} \cos(2t) + \frac{\langle \sin^2(t), \sin(2t) \rangle}{\langle \sin(2t), \sin(2t) \rangle} \sin(2t) \\ &= \frac{1}{2}(1 - \cos(2t)). \end{aligned}$$

Note that we actually have equality here (by a trig identity), so it turns out that $\sin^2(t)$ is in fact already contained in the space we called W_2 in class, even though it didn't necessarily appear to be.

4. Use properties of inner products to verify the parallelogram law:

$$2 \| \mathbf{v} \|^2 + 2 \| \mathbf{w} \|^2 = \| \mathbf{v} + \mathbf{w} \|^2 + \| \mathbf{v} - \mathbf{w} \|^2 .$$

Let's start with the right side. We have

$$\begin{aligned} \| \mathbf{v} + \mathbf{w} \|^2 + \| \mathbf{v} - \mathbf{w} \|^2 &= \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w} \rangle + \langle \mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{w} \rangle \\ (\langle \mathbf{v}, \mathbf{v} \rangle + 2\langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle) &+ (\langle \mathbf{v}, \mathbf{v} \rangle - 2\langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle) \\ &= 2\langle \mathbf{v}, \mathbf{v} \rangle + 2\langle \mathbf{w}, \mathbf{w} \rangle = 2 \| \mathbf{v} \|^2 + 2 \| \mathbf{w} \|^2 . \end{aligned}$$