Calculus I Mini-test 2 Solutions July 3, 2008

1. Find the absolute maximum and minimum values of the function $f(x) = (x^2 + 2x)^3$ on the interval $[-2, 1]$.

We apply the extreme value theorem. Using the chain rule, we find the derivative:

$$f'(x) = 3(x^2 + 2x)^2(2x + 2).$$

Setting this equal to zero and solving gives

$$2x + 2 = 0 \quad \Leftrightarrow \quad x = -1,$$

or

$$x^2 + 2x = x(x + 2) = 0 \quad \Leftrightarrow \quad x = 0 \text{ or } x = -2.$$

These are the three critical points for this function, so each of these is a potential max or min. Plugging in we find

$$f(0) = 0 \qquad f(-1) = -1 \qquad f(-2) = 0.$$

Finally, we check the endpoints, for which we have

$$f(-2) = 0 \qquad f(1) = 27.$$

Thus the absolute maximum on this interval is 27 and occurs at $x = 1$, while the absolute minimum is -1 and occurs at $x = -1$.

2. Show that the function $f(x) = x^{101} + x^{51} + x - 1$ has exactly one root. Be sure to clearly state any theorems you use.

The derivative is

$$f'(x) = 101x^{100} + 51x^{50} + 1.$$

Note that this derivative is a sum of even powers of x, plus one. Thus $f'(x) \geq 1$ for all x. In particular, $f'(x)$ is never equal to zero. By Rolle's theorem, the fact that $f'(x)$ has no roots implies that $f(x)$ has at most one root.

To see that it does, in fact, actually have this one root, we note that $f(1) = 2 > 0$, while $f(0) = -1$. The intermediate value theorem then implies that $f(x) = 0$ for some x in between 1 and 0.

Rolle's Theorem: If $f(x)$ is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then $f'(c) = 0$ for some $a < c < b$. (We're using it in its contrapositive form, which says that if $f'(x)$ has n roots on $[a, b]$, then $f(x)$ has at most $n + 1$ roots on $[a, b]$.)

Intermediate Value Theorem: If $f(x)$ is continuous on $[a, b]$, $f(a) < 0$ and $f(b) > 0$, then $f(c) = 0$ for some $a < c < b$. (It's really more general than this, but this is the application of it we use here.)

3. Suppose $f(0) = 1$ and $2 \leq f'(x) \leq 5$ for all x in $[0, 4]$. Use the Mean Value Theorem to show that $9 \leq f(4) \leq 21$.

Applying the Mean Value Theorem, we see that

$$f'(c) = \frac{f(4) - f(0)}{4 - 0} = \frac{f(4) - 1}{4}$$

for some c between a and b. But we know that $2 \leq f'(x) \leq 5$ for all x; in particular, this inequality holds for $x = c$. Applying these inequalities to $f'(c)$ gives

$$2 \leq f'(c) \leq 5 \quad \Leftrightarrow \quad 2 \leq \frac{f(4) - 1}{4} \leq 5.$$

Solving for $f(4)$, we obtain

$$8 \leq f(4) - 1 \leq 20 \qquad \Leftrightarrow \qquad 9 \leq f(4) \leq 21.$$

Mean Value Theorem: If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ for some $a < c < b$.