Test 1 Solutions

Calculus I **June 20, 2008**

1. Find the equation for the line tangent to the graph of $y = 2x^3 - 5x$ at $(2, 6)$.

   The derivative at $x = 2$ gives the slope of the tangent line. We have $\frac{dy}{dx} = 6x^2 - 5$, which at $x = 2$ is 19. Thus the line is given by

$$y - 6 = 19(x - 2) \qquad \text{or} \qquad y = 19x - 32.$$

2. (a) Use the definition of the derivative to prove that $\dfrac{d}{dx}(af(x)) = a\dfrac{df}{dx}$.

$$\frac{d}{dx}(af(x)) = \lim_{h\to 0} \frac{af(x+h) - af(x)}{h} = a \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = a\frac{df}{dx}.$$

(b) Use the product and chain rule to differentiate $f(x)\big(g(x)\big)^{-1}$ (what you get in the end will be equivalent to the quotient rule).

   We apply the product rule first, obtaining

$$f'(x)(g(x))^{-1} + f(x)[(g(x))^{-1}]'.$$

   Applying the chain rule to the last term, we find

$$[(g(x))^{-1}]' = -1[g(x)]^{-2}g'(x),$$

   so that our answer is

$$f'(x)(g(x))^{-1} - f(x)(g(x))^{-2}g'(x).$$

   Note that if we get a common denominator of $(g(x))^2$, this equals

$$\frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

3. Differentiate the following:

(a) $f(x) = \ln(\sec(x))$

   This is a composition of two functions: we may write it as $f(u) = \ln(u)$, where $u = \sec(x)$. Thus by the chain rule we have

$$f'(x) = \frac{1}{u}\frac{du}{dx} = \frac{\sec(x)\tan(x)}{\sec(x)} = \tan(x).$$

(b) $f(x) = \dfrac{\sqrt{x+1}}{x}$

Using the quotient rule (and noting that $\sqrt{x+1} = (x+1)^{1/2}$), we have

$$f'(x) = \frac{(x)'(x+1)^{1/2} - (x)[(x+1)^{1/2}]'}{x^2}.$$

To differentiate $(x+1)^{1/2}$, we use the chain rule, obtaining $\frac{1}{2}(x+1)^{-1/2}(1)$, so that the final answer is

$$f'(x) = \frac{(x+1)^{1/2} - \frac{x}{2}(x+1)^{-1/2}}{x^2}.$$

(c) $f(x) = x^2 \sin(3x)$

Using the product rule, we have

$$f'(x) = (x^2)' \sin(3x) + x^2(\sin(3x))'.$$

For the derivative of the sine function, we apply the chain rule, obtaining

$$(sin(3x))' = 3\cos(3x),$$

so that the final answer is

$$f'(x) = 2x\sin(3x) + 3x^2\cos(3x).$$

4. Suppose $g(x) = (h(x))^3$. Find $g'(0)$, given that $h(0) = -\frac{1}{2}$ and $h'(0) = \frac{8}{3}$

By the chain rule, we have

$$g'(x) = 3(h(x))^2 h'(x).$$

Plugging in $x = 0$ we get

$$g'(0) = 3(h(0))^2 h'(0) = 3(-1/2)^2(8/3) = 2.$$

5. Use the squeeze theorem to calculate the following limit: $\lim_{x\to 0} \sqrt{x^3 + x^2}\sin(\pi/x)$.

First note that we cannot just plug in $x = 0$, because π/x then becomes undefined. Using the fact that $-1 \le \sin(t) \le 1$ for any value of t, we have

$$-\sqrt{x^3 + x^2} \le \sqrt{x^3 + x^2}\sin(\pi/x) \le \sqrt{x^3 + x^2}.$$

As $x \to 0$, the outer expressions both go to zero (you can just plug in $x = 0$ to these), so the squeeze theorem implies that the center expression also goes to zero.

6. Briefly explain why each of the following is false (in particular, say what the correct answer is):

(a) $\frac{d}{dx}(10^x) = x10^{x-1}$

`This is applying the power rule to an exponential. It should be` $\frac{d}{dx}10^x = \ln(10)10^x$.

(b) $\frac{d}{dx}(e^2) = 2e$

`Because` e^2 `is a constant, its derivative is zero.`

(c) $\frac{d}{dx}(e^2) = e^2$

`Because` e^2 `is a constant, its derivative is zero.`

(d) $[f(x)g(x)]' = f'(x)g'(x)$

`Products do not differentiate this way. It should be` $[f(x)g(x)]' = f'(x)g(x) + g'(x)f(x)$.