1. Show that $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$, where

$$C: \quad \mathbf{r}(t) = \langle e^{t\sin(\pi t)}, t - t^5\rangle, \qquad 0 \le t \le 1$$
$$\mathbf{F}(x,y) = \langle 2e^{2x}\sin y - e^{x^2},\ e^{2x}\cos y + \ln(y^2+1)\rangle.$$

(Hint: You'll need to check that $\mathbf{F}$ has a certain property *and* that C has a certain property.)

We check that $\mathbf{F}$ is conservative (note that $\mathbf{F}$ is defined everywhere, so we don't have to worry about things being simply-connected):

$$\frac{\partial}{\partial y}(2e^{2x}\sin y - e^{x^2}) = 2e^{2x}\cos y = \frac{\partial}{\partial x}(e^{2x}\cos y + \ln(y^2+1),$$

which verifies that $\mathbf{F}$ is conservative.

Also, we have

$$\mathbf{r}(1) = \langle e^0, 0-0\rangle = \langle 1,0\rangle = \langle e^0, 0-0\rangle = \mathbf{r}(1),$$

so that C is a closed loop. Since conservative vector fields integrate to zero on closed loops, this integral must be zero.

2. Use the Fundamental Theorem for Line Integrals to compute $\int_C \nabla f\cdot d\mathbf{r}$, where

$$f(x,y) = e^y\sin(2x) + x^2y$$
$$C: \quad \mathbf{r}(t) = \langle t\cos^2 t, e^{t^2\sin t}\rangle, \qquad 0 \le t \le \pi.$$

(No integrating or differentiating, please.)

The FTLI says

$$\int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Note that $\mathbf{r}(\pi) = \langle \pi, 1\rangle$ and $\mathbf{r}(0) = \langle 0,1\rangle$, so

$$\int_C \nabla f\cdot d\mathbf{r} = f(\pi,1) - f(0,1) = \pi^2 - 0 = \pi^2.$$

Bonus:

$$\mathbf{F} \text{ is conservative on } D \quad \Leftrightarrow \quad \int_C \mathbf{F}\cdot d\mathbf{r} \text{ is path}-\text{independent on } D$$
$$\Leftrightarrow \quad \int_C \mathbf{F}\cdot d\mathbf{r} = 0 \text{ on all closed loops in } D \quad \Leftrightarrow \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ in } D \subseteq \mathbf{R}^2.$$