Line Integrals

1. Integrating a function along a curve:

$$\int_C f\,ds = \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt.$$

2. Integrating a vector field along a curve:
   (a) In general, evaluate as

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt.$$

(b) If C is closed (i.e., $\mathbf{r}(a) = \mathbf{r}(b)$), try Green's/Stokes' theorem. Find a surface S with C as boundary and use the formula

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\operatorname{curl}\mathbf{F})\cdot d\mathbf{S}.$$

If you're in the plane and want the normal component, you may use the divergence form of Green's theorem:

$$\int_C (\mathbf{F}\cdot\mathbf{n})\,ds = \iint_D \operatorname{div}\mathbf{F}\,dA.$$

(c) If $\mathbf{F}$ is conservative (i.e., $\operatorname{curl}\mathbf{F} = \mathbf{0}$), use the fundamental theorem for line integrals. Either replace C with any other path having the same endpoints and evaluate directly, as in (a), or find a potential function f for $\mathbf{F}$ and use

$$\int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

(d) If $\mathbf{F}$ is conservative *and* C is closed, the answer is zero (by the fundamental theorem for line integrals or Stokes' theorem).

Surface Integrals

1. Integrating a function over a surface:

$$\iint_S f\,dS = \iint_D f(\mathbf{r}(u,v))\,|\mathbf{r}_u \times \mathbf{r}_v|\,dA.$$

2. Integrating a vector field over a surface:
   (a) In general, evaluate as

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA.$$

(b) If S is inflatable, try the divergence theorem. Find a solid E with S as boundary and use the formula

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV.$$

(c) If $\mathbf{F}$ is a curl (i.e., $\operatorname{div}\mathbf{F} = 0$), use Stokes' theorem. Either replace S with any other surface having the same boundary curve C and evaluate directly, as in (a), or find a vector field $\mathbf{G}$ so that $\mathbf{F} = \operatorname{curl}\mathbf{G}$ and use

$$\iint_S (\operatorname{curl}\mathbf{G})\cdot d\mathbf{S} = \int_C \mathbf{G}\cdot d\mathbf{r}.$$

(d) If $\mathbf{F}$ is a curl *and* S is closed, the answer is zero (by Stokes' theorem or by the divergence theorem).