Calculus III Mini-test 2 Solutions July 3, 2008

1. Compute $\iint_D x\,dA$, where D is the region in the xy-plane bounded by the curve $y = x^2$ and the line $y = x + 2$.

The region is y-simple; i.e., vertical lines hit the region nicely. Thus it should be described as

$$g_1(x) \le y \le g_2(x) \qquad a \le x \le b.$$

The curve on which the tops of vertical segments in D lie is $y = x + 2$, while that on which the bottoms lie is $y = x^2$. The x-values where the two curves intersect are solutions to $x^2 = x + 2$; i.e., $x = -1$ and $x = 2$. Thus the region is given by

$$x^2 \le y \le x + 2 \qquad -1 \le x \le 2.$$

The integral should therefore be set up as follows:

$$\iint_D x\,dA = \int_{-1}^{2}\int_{x^2}^{x+2} x\,dy\,dx = \int_{-1}^{2} [xy]_{y=x^2}^{y=x+2}\,dx$$

$$= \int_{-1}^{2} (x^2+2x-x^3)\,dx = \left[\frac{x^3}{3} + x^2 - \frac{x^4}{4}\right]_{x=-1}^{x=2} = \left(\frac{8}{3}+4-4\right)-\left(\frac{-1}{3}+1-\frac{1}{4}\right) = \frac{9}{4}.$$

2. A thin metal sheet occupies the region D in the xy-plane bounded by the parabolas $x = 2y^2$ and $x = 1 + y^2$ and lying above the x-axis. The density of the sheet at the point (x, y) is described by $f(x, y) = y + 2$. Find the mass of the sheet by integrating the density function over D.

The region is x-simple; i.e., horizontal lines hit the region nicely. Thus it should be described as

$$h_1(y) \le x \le h_2(y) \qquad c \le y \le d.$$

The curve on which the right sides of horizontal segments in D lie is $x = y^2+1$, while that on which left sides lie is $x = 2y^2$. The top y-value where the two curves intersect is the positive solution to $2y^2 = y^2 + 1$ (remember we're only considering the part of the region that lies above the x-axis). Thus y ranges from 0 to 1, so D is described by

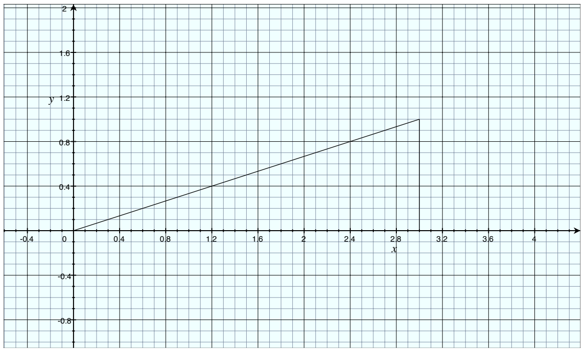
$$2y^2 \le x \le y^2 + 1 \qquad 0 \le y \le 1.$$

The integral should therefore be set up as follows:

$$\iint_D (y+2)\,dA = \int_0^1\int_{2y^2}^{y^2+1} (y+2)\,dx\,dy = \int_0^1 [x(y+2)]_{2y^2}^{y^2+1}\,dy$$

$$= \int_0^1 -y^3 - 2y^2 + y + 2\,dy = \left[\frac{-y^4}{4} - \frac{2y^3}{3} + \frac{y^2}{2} + 2y\right]_0^1 = \frac{19}{12}.$$

3. Carefully sketch the region of integration for the integral $\int_0^1\int_{3y}^{3} e^{x^2}\,dx\,dy$. Then reverse the order of integration and evaluate the integral.



These limits of integration correspond to inequalities of the following form:

$$3y \leq x \leq 3 \qquad 0 \leq y \leq 1.$$

Graphing the equations $x = 3y$, $x = 3$, $y = 0$, and $y = 1$, we see that the region looks as shown in the figure. Switching the order of integration means treating this region as a y-simple region, for which we get inequalities of the form

$$0 \leq y \leq x/3 \qquad 0 \leq x \leq 3.$$

We now have

$$\int_0^3 \int_0^{x/3} e^{x^2}\,dy\,dx = \int_0^3 \left[ye^{x^2}\right]_0^{x/3} dx = \frac{1}{3}\int_0^3 xe^{x^2}\,dx$$
$$= \frac{1}{6}\left[e^{x^2}\right]_0^3 = \frac{1}{6}(e^9 - 1),$$

where we use u-substitution for the final integral, with $u = x^2$ and $du = 2x\,dx$ or $x\,dx = \frac{1}{2}du$.