INTRODUCTION

If C is a curve and f is a function defined at all points of C, then the integral of f over C does the following (roughly speaking):

- Chop C into small pieces each of which has size Δs;
- For each piece C_i of C, pick a point p_i; then $f(p_i)$ is some number;
- Add up the products $f(p_i)\Delta s$ for each i;
- Take the limit as $\Delta s \to 0$, obtaining $\int_C f\, ds$.

When we say below that an integral is "adding up number along/across/over C," we mean to refer to this process as described above.

PATH INTEGRALS

An integral $\int_C f\, ds$ of a function f over a curve C simply adds up numbers along the curve, where the numbers are given by the function f. To calculate such an integral, we need to have a parametrization $\mathbf{r}(t)$ $(a \leq t \leq b)$ of the curve C. We then compute using the following:

$$\int_C f\, ds = \int_a^b f\big(\mathbf{r}(t)\big)\, |\mathbf{r}'(t)|\, dt.$$

Special Cases:

- When C is a piece of the x-axis, say between $x = a$ and $x = b$, we use x as the parameter, and the expression above becomes

  $$\int_C f\, ds = \int_a^b f(x)\, dx.$$

  One often takes the function f to represent height (measured on the y-axis), so that this integral gives (signed) area under the graph of $y = f(x)$.

- When the function f is identically 1, this integral simply adds up all the Δs bits, and hence gives the *arclength* of C. Thus we have

  $$\text{Length}(C) = \int_C ds = \int_a^b |\mathbf{r}'(t)|\, dt.$$

- When X is a curve in the plane, note that we have

  $$|\mathbf{r}'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

  One often takes the function f to represent height (measured on the z-axis), so that this integral gives (signed) area of a curtain hanging from the curve $f(C)$ down to the xy-plane.

- When C is a curve in space, we have

$$|\mathbf{r}'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}.$$

- In the presence of a vector field $\mathbf{F}$ in the plane or space, we often consider the function f defined as the tangential component of $\mathbf{F}$ along C; i.e., $f = \mathbf{F} \cdot \mathbf{T}$, where $\mathbf{T}$ is the unit tangent vector $\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ to the curve C, so that the integral along C becomes

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \left(\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right) |\mathbf{r}'(t)|\, dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt,$$

where the left-most expression is just short-hand notation for the case where this particular function is the one being integrated. This is such a useful thing to do that we simply call it "integrating the vector field along the curve." It is important to remember that we're still just integrating a function over a curve. It's just that when we say we are "integrating a vector field," we have a specific function in mind, namely, the one giving the tangential component of the vector field along the curve. When you write this integral out in terms of the components $\mathbf{F} = \langle P, Q, R\rangle$ and $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, you obtain the expression (in two dimension, eliminate all terms with z or R):

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\right) dt = \int_C P\, dx + Q\, dy + R\, dz,$$

where the right-most expression is just shorthand notation for the integral in the middle.

- When the vector field $\mathbf{F}$ is in the plane, we can also consider the function f defined as the normal component of $\mathbf{F}$ along C; i.e., $f = \mathbf{F} \cdot \mathbf{n}$, where $\mathbf{n}$ is the unit normal vector to the curve C, so that the integral of this function along C becomes

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C \left(P\frac{dy}{dt} - Q\frac{dx}{dt}\right) dt = \int_C P\, dy - Q\, dx,$$

where, again, the right-most expression is just shorthand notation for the integral in the middle. (This doesn't make sense for curves in space because there are lots of normal directions to a curve in space; see the next section for 3D flux.)