INTRODUCTION

If S is a surface and f is a function defined at all points of S, then the integral of f over S does the following (roughly speaking):

- Chop S into small 'rectangular' pieces, where the ith piece has area ΔS_i;
- For each piece S_i of S, pick a point p_i; then $f(p_i)$ is some number;
- Add up the products $f(p_i)\Delta S_i$ over all i;
- Take the limit as $\Delta S_i \to 0$. obtaining $\iint_S f\,dS$.

This is the procedure to which we refer when we say an integral "adds up numbers over/across S."

SURFACE INTEGRALS

An integral $\iint_S f\,dS$ of a function f over a surface S simply adds up numbers along the surface, where the numbers are given by the function f. To calculate such an integral, we need to have a parametrization $\mathbf{r}(u,v)$ $((u,v) \in D)$ of the surface S. We then compute using the following:

$$\iint_C f\,dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,dA.$$

Special Cases:

- When S is some region D of the xy-plane, we can use parameters $u = x$ and $v = y$, and the expression becomes
$$\iint_S f\,dS = \iint_D f(x,y)\,dA.$$
One often takes the function f to represent height (measured via the z-axis), so that this integral gives (signed) volume under the graph $z = f(x,y)$. This is the usual double integral from earlier this semester.

- When the function f is identically 1, this integral simply adds up all the ΔS bits, and hence gives the *surface area* of S. Thus we have
$$\text{Area}(S) = \iint_S dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,dA.$$

- When S is the graph of a function of the form $z = f(x,y)$, note that we have
$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}.$$

- In the presence of a vector field $\mathbf{F}$ in space, we often consider the function f defined as the normal component of $\mathbf{F}$ across S; i.e., $f = \mathbf{F} \cdot \mathbf{n}$, where $\mathbf{n}$ is the unit normal vector $\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$ to the surface S, so that the integral of this function across S becomes

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(\mathbf{F}(\mathbf{r}(u,v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \right) |\mathbf{r}_u \times \mathbf{r}_v| \, dA = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA,$$

where the left-most expression is just shorthand notation for the case that this particular function is the one being integrated. This is such a useful thing to do that we simply call it "integrating the vector field across the surface," and the quantity thus obtained is called the *flux* of the vector field across the surface.

> *Note*: For this to be defined, the surface needs to be orientable, so that there is a consistent choice of normal vector onto which to project. Roughly speaking, you need to be able to color one side of the surface red and the other side blue, without ever accidentally coloring anything purple.

Comments

- Our surface parametrizations usually come from representing our surface as the graph of a function in some standard coordinate system. Thus it is helpful to memorize the result for $|\mathbf{r}_u \times \mathbf{r}_v|$ in each of these coordinates. Thus for spherical ($\rho = f(\theta, \phi)$), cylindrical ($r = f(\theta, z)$), and cartesian ($z = f(x, y)$), we have

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \rho^2 \sin\phi \qquad |\mathbf{r}_\theta \times \mathbf{r}_z| = r \qquad |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{(f_x)^2 + (f_y)^2 + 1}.$$

- To evaluate an "ordinary" double integral $\iint_D f(x,y)\, dA$ over a region D that is most conveniently described in some other coordinate system (polar, for instance), we imagine D as a surface in space (with $z = 0$), parametrize it using these new coordinates, and evaluate it as a surface integral.