# Test 1 Solutions

**Calculus III** **June 20, 2008**

1. Consider the function $\mathbf{r}(t) = \langle \cos(t), t, \sin(t) \rangle$.

(a) Find the curvature $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$ at the point $(1,0,0)$.

We first compute

$$\mathbf{r}'(t) = \langle -\sin t, 1, \cos t \rangle \qquad |\mathbf{r}'(t)| = \sqrt{\sin^2 t + 1 + \cos^2 t} = \sqrt{2},$$

so

$$\mathbf{T}(t) = \frac{1}{\sqrt{2}}\langle -\sin t, 1, \cos t \rangle \qquad \mathbf{T}'(t) = \frac{1}{\sqrt{2}}\langle -\cos t, 0, -\sin t \rangle,$$

and

$$|\mathbf{T}'(t)| = \frac{1}{\sqrt{2}}\sqrt{\cos^2 t + \sin^2 t} = \frac{1}{\sqrt{2}}.$$

It follows that

$$\kappa(t) = \frac{1}{2}$$

for all $t$ (in particular for $t = 0$, which puts us at the desired point).

(b) Find a parametrization of the line tangent to the curve above at the same point, $(1,0,0)$.

We want to parametrize the line through the point $(1,0,0)$ in the direction of $\mathbf{r}'(0) = \langle 0, 1, 1 \rangle$ (this vector is always tangent to the curve parametrized by $\mathbf{r}(t)$). Thus we have

$$\mathbf{r}(t) = \langle 1, 0, 0 \rangle + t\langle 0, 1, 1 \rangle = \langle 1, t, t \rangle.$$

2. Consider the function $f(x,y,z) = x^2y + y^2x - z^2x$

(a) Compute the gradient vector field $\nabla f$.

We compute

$$\nabla f(x,y,z) = \langle 2xy + y^2 - z^2, x^2 + 2xy, -2xz \rangle.$$

(b) Compute the directional derivative of $f$ in the direction of the vector $\mathbf{v} = \langle 1, 1, 1 \rangle$ at the point $(1,0,2)$.

We compute

$$\nabla f(1,0,2) = \langle -4, 1, -4 \rangle.$$

Dotting this with the unit-length version of $\mathbf{v}$, we get

$$\langle -4, 1, -4 \rangle \cdot \frac{1}{\sqrt{3}}\langle 1, 1, 1 \rangle = \frac{-7}{\sqrt{3}}.$$

3. Suppose $f$ has continuous second partial derivatives. Show that the curl of its gradient vector field is zero. (Hint: The vector field has the form $\mathbf{F} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$; compute the curl; explain why each component is zero.)

`Computing the curl directly, we obtain`

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} = \left\langle \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right\rangle.$$

`Each entry of this vector is zero by the equality of mixed partials (Clairaut's theorem).`

4. Consider the function $f(x, y) = x^2 + y^2 + kxy$.

(a) Show that $(0, 0)$ is always a critical point for this function, regardless of the value of $k$.

`The first derivatives at` $(0,0)$ `are`

$$f_x(0,0) = 2x + ky|_{(0,0)} = 0 \qquad \text{and} \qquad f_y(0,0) = 2y + kx|_{(0,0)} = 0.$$

`Thus` $(0,0)$ `is a critical point, regardless of the value of` $k$.

(b) Classify this critical point for all possible values of $k$, being sure to indicate for which $k$ the second derivative test fails.

`We compute` $D = f_{xx}f_{yy} - [f_{xy}]^2$, `finding`

$$f_{xx}(0,0) = 2 \qquad f_{yy}(0,0) = 2 \qquad f_{xy}(0,0) = k.$$

`Thus` $D = 4 - k^2$. `When` $k = \pm 2$, `we have that` $D = 0$, `so that the second derivative test fails. When` $-2 < k < 2$, `we have that` $D > 0$, `so the point is either a max or min. Because` $f_{xx} > 0$, `it must be a min. When` $|k| > 2$, `we have that` $D < 0$, `so the point is a saddle.`

(c) Write an explicit system of equations that can be used to optimize this function subject to the constraint $x^2 + y^2 = 1$. Do not solve the system.

`Using` $g(x, y) = x^2 + y^2 - 1$, `we set` $\nabla f = \lambda \nabla g$ `(and throw in the constraint equation) to get`

$$\begin{cases} 2x + ky = 2\lambda x \\ 2y + kx = 2\lambda y \\ x^2 + y^2 = 1 \end{cases}$$

5. Consider the sphere of radius $\sqrt{2}$ centered at the origin, given by $x^2 + y^2 + z^2 = 2$.

(a) Parametrize this surface.

Using spherical coordinates, we have

$$\langle x,y,z\rangle = \langle \rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi\rangle = \langle \sqrt{2}\cos\theta\sin\phi, \sqrt{2}\sin\theta\sin\phi, \sqrt{2}\cos\phi\rangle,$$

so this gives a parametrization of the surface as a function of $\phi$ and $\theta$.

(b) Find an equation for the plane tangent to this surface at the point $(x,y,z)=(1,0,1)$. (Hint: At this point, $\theta = 0$.)

We first find partial derivatives:

$$\mathbf{r}_\theta = \langle -\sqrt{2}\sin\theta\sin\phi, \sqrt{2}\cos\theta\sin\phi, 0\rangle \qquad \mathbf{r}_\phi = \langle \sqrt{2}\cos\theta\cos\phi, \sqrt{2}\sin\theta\cos\phi, -\sqrt{2}\sin\phi\rangle.$$

Plugging in $\theta = 0$ and $\phi = \pi/4$, these become

$$\mathbf{r}_\theta = \langle 0,1,0\rangle \qquad \mathbf{r}_\phi = \langle 1,0,-1\rangle.$$

These describe two vectors tangent to the sphere at our point, so their cross product will be normal. Crossing, we have

$$\mathbf{r}_\theta \times \mathbf{r}_\phi = \langle -1,0,-1\rangle.$$

Thus our tangent plane is given by

$$-1(x-1)+0(y-0)-1(z-1)=0 \qquad \text{or} \qquad x+z=2.$$

6. If $x$ and $y$ are the lengths of two sides of a triangle, and the angle they contain is $\theta$, then the area of the triangle is given by $A = \frac{1}{2}xy\sin\theta$. Now suppose $x$, $y$, and $\theta$ are all changing, so that they are functions of $t$.

(a) Use the chain rule to find a formula for the derivative $\dfrac{dA}{dt}$.

Note that $A$ is a function of $x$, $y$, and $\theta$, each of which is a function of $t$. Thus we have

$$\frac{dA}{dt} = \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt} + \frac{\partial A}{\partial \theta}\frac{d\theta}{dt}.$$

(b) Suppose at time $t=0$ our triangle has dimensions given by $x = 24$ in., $y = 20$ in., and $\theta = \pi/6$. Suppose further that, at time $t = 0$, $x$ is increasing at a rate of 2 in/s, $y$ is decreasing at a rate of 3 in/s, and $\theta$ is increasing at a rate of $\frac{1}{20}$ rad/s. Is the area of the triangle increasing or decreasing at time $t = 0$? At what rate? (Hint: You'll need to compute things like $\partial A/\partial x$ explicitly from the formula above and plug in the given values; for things like $\partial x/\partial t$, use the information given.)

Note that

$$\frac{\partial A}{\partial x} = \frac{1}{2}y\sin\theta \qquad \frac{\partial A}{\partial y} = \frac{1}{2}x\sin\theta \qquad \frac{\partial A}{\partial \theta} = \frac{1}{2}xy\cos\theta.$$

Evaluating these with $x=24$, $y=20$, and $\theta=\pi/6$, we have

$$\frac{\partial A}{\partial x} = 5 \qquad \frac{\partial A}{\partial y} = 6 \qquad \frac{\partial A}{\partial \theta} = 120\sqrt{3}.$$

Plugging these values in to the chain rule above, along with $\frac{dx}{dt} = 2$, $\frac{dy}{dt} = -3$, and $\frac{d\theta}{dt} = \frac{1}{20}$, we have

$$\frac{dA}{dt}|_{t=0} = (5)(2)+(6)(-3)+(120\sqrt{3})(1/20) = 6\sqrt{3}-8.$$

This is positive, so the area of the triangle is increasing at this rate.