

1. Compute $\int_C 8e^y ds$, where C is given by $\mathbf{r}(t) = \langle t, \ln(\sec t) \rangle$ for $0 \leq t \leq \frac{\pi}{3}$.

The formula is

$$\int_C f ds = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| dt.$$

Here we have $f(\mathbf{r}(t)) = 8e^{\ln(\sec t)} = 8 \sec t$. Note that $\frac{d}{dt}(\ln(\sec t)) = \frac{\sec t \tan t}{\sec t} = \tan t$, so that $|\mathbf{r}'(t)| = \sqrt{1^2 + \tan^2 t} = \sec t$. Thus we have

$$\int_C 8e^y ds = 8 \int_0^{\pi/3} \sec^2 t dt = 8 \tan t \Big|_0^{\pi/3} = 8\sqrt{3}.$$

2. Use the fundamental theorem for line integrals to verify that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$, where C is given by $\mathbf{r}(t) = \langle t^4 - t^3, t \sin(\pi t) \rangle$ for $0 \leq t \leq 1$, and where

$$\mathbf{F}(x, y) = \langle 2e^y \cos(2x) + 2xy, e^y \sin(2x) + x^2 \rangle.$$

To use the fundamental theorem for line integrals, we need to verify that \mathbf{F} is conservative. To this end we note that

$$\frac{\partial Q}{\partial x} = 2e^y \cos(2x) + 2x = \frac{\partial P}{\partial y}.$$

Further, note that

$$\mathbf{r}(0) = \langle 0, 0 \rangle = \mathbf{r}(1),$$

so that C is a closed curve. The fundamental theorem of line integrals implies that integrals of conservative vector fields along closed curves are zero.

3. Express the area of the surface $\mathbf{r}(u, v) = \langle u \cos v, u \sin v, 2v \rangle$, with domain D given by $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$, as a double integral in u and v . Do not evaluate the integral.

In general, the formula is $\iint_S f dS = \iint_D f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| dA$. To compute surface area, we choose $f = 1$. We compute

$$\begin{aligned} |\mathbf{r}_u \times \mathbf{r}_v| &= |\langle \cos v, \sin v, 0 \rangle \times \langle -u \sin v, u \cos v, 2 \rangle| \\ &= |\langle 2 \sin v, -2 \cos v, u \cos^2 v + u \sin^2 v \rangle| = |\langle 2 \sin v, -2 \cos v, u \rangle| = \sqrt{4 + u^2}. \end{aligned}$$

Thus we have

$$\text{Area} = \int_0^1 \int_0^{2\pi} \sqrt{4 + u^2} dv du.$$

4. Use Green's theorem to compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle xy^2, -x^2y \rangle$, and where C consists of the part of the parabola $y = x^2$ between $(-1, 1)$ and $(1, 1)$, along with the straight line joining $(1, 1)$ to $(-1, 1)$.

Green's theorem says that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA,$$

where D is the region bounded by C . Here we have $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -2xy - 2xy = -4xy$. Also, the region D is y -simple, and can be described by $x^2 \leq y \leq 1$ with $-1 \leq x \leq 1$. Thus we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -4 \int_{-1}^1 \int_{x^2}^1 xy \, dy \, dx = -4 \int_{-1}^1 \left(x - \frac{x^5}{2} \right) dx = 0.$$

5. Use Stokes' theorem to compute $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle 1, xy^2, xy^2 \rangle$ and S is the part of the plane $y + z = 2$ inside the cylinder $x^2 + y^2 = 1$.

Stokes' theorem says that

$$\iint_S \text{curl } \mathbf{F} \, d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt.$$

(The second equality is just the formula for integrating vector fields along curves.) The boundary of S is a curve C running around the cylinder $x^2 + y^2 = 1$ and lying on the plane $y + z = 2$. Together, these facts imply that we may parametrize the curve via $\mathbf{r}(t) = \langle \cos t, \sin t, 2 - \sin t \rangle$. Then we have

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle 1, \cos t \sin^2 t, \cos t \sin^2 t \rangle \cdot \langle -\sin t, \cos t, -\cos t \rangle dt \\ &= - \int_0^{2\pi} \sin t \, dt = 0. \end{aligned}$$

6. Verify that the divergence theorem is true for the vector field $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$ where E is the unit ball $x^2 + y^2 + z^2 \leq 1$.

(Hint: For the usual parametrization of the unit sphere $\mathbf{r}(\phi, \theta) = \langle \cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi \rangle$, we have that $\mathbf{r}_\phi \times \mathbf{r}_\theta = \langle \cos \theta \sin^2 \phi, \sin \theta \sin^2 \phi, \sin \phi \cos \phi \rangle$.)

The divergence theorem states that

$$\iiint_E \text{div } \mathbf{F} \, dV = \iint_{\partial E} \mathbf{F} \cdot d\mathbf{S}.$$

Computing the double integral, we have

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F}(\mathbf{r}(\phi, \theta)) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) \, dA \\ &= \int_0^{2\pi} \int_0^\pi \langle \cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi \rangle \cdot \langle \cos \theta \sin^2 \phi, \sin \theta \sin^2 \phi, \sin \phi \cos \phi \rangle \, d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^\pi \sin \phi \, d\phi \, d\theta = 4\pi. \end{aligned}$$

For the triple integral, note that $\text{div } \mathbf{F} = 1 + 1 + 1 = 3$, so that

$$\iiint_E \text{div } \mathbf{F} \, dV = 3 \iiint_E dV = 3 \text{vol}(E) = 4\pi.$$

This verifies the divergence theorem.