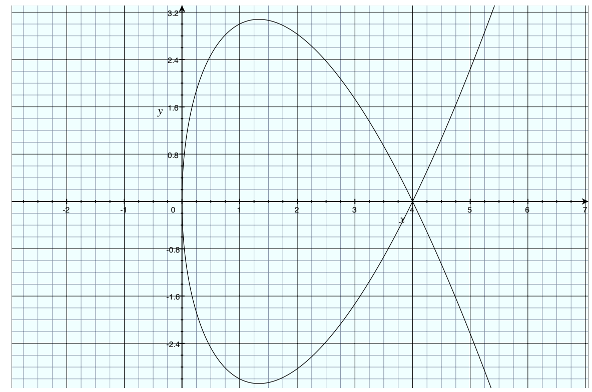
Calculus II Test 2 Solutions July 17, 2009

1. Consider the parametrized curve $c(t) = (t^2,\ t^3 - 4t)$ shown below.
(a) Find all values of t for which the curve crosses the x-axis.



Points where the graph crosses the x-axis are points where $y = 0$. Thus we set $t^3 - 4t = 0$ and solve, obtaining $t(t+2)(t-2) = 0$, so $t = 0, \pm 2$.

(b) Set up, but do not evaluate, an integral (in t) equal to the length of the arch lying above the x-axis.

We use the formula for parametric arc length. For this we compute

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (2t)^2 + (3t^2 - 4)^2.$$

Combining this with the information we got in part (a), we obtain the integral

$$L = \int_{-2}^{0} \sqrt{4t^2 + (3t^2 - 4)^2}\, dt.$$

(I've integrated from -2 to 0, because those are the values of t that span the upper arch; integrating from 0 to 2 gives the same value, however, and will be counted as correct.)

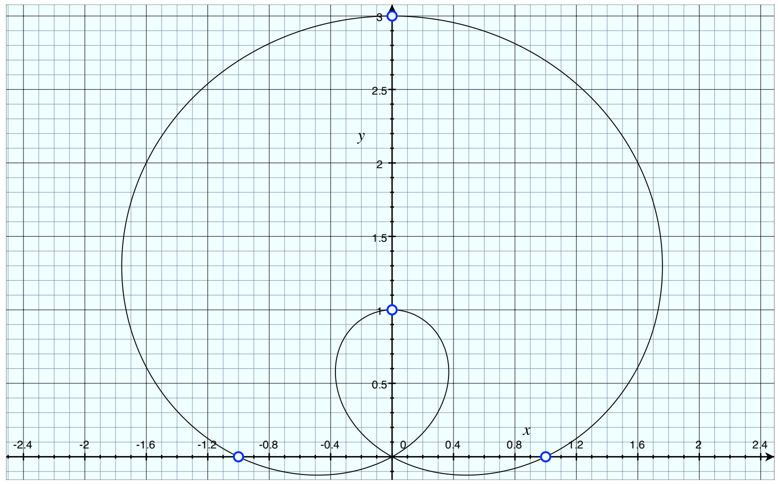
2. Consider the polar curve $r = 1 + 2\sin\theta$ shown below.

(a) On the picture, label the points corresponding to the values $\theta = 0$, $\pi/2$, π, and $3\pi/2$.

(b) Find a Cartesian parametrization of the curve.

We use the equations relating polar and Cartesian coordinates:

$$x = r\cos\theta = (1 + 2\sin\theta)\cos\theta \qquad y = r\sin\theta = (1 + 2\sin\theta)\sin\theta.$$



(c) Find the slope of the line tangent to the graph at the point where $\theta = 0$.

We use the parametric equations above, computing

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos\theta\sin\theta + (1+2\sin\theta)\cos\theta}{2\cos\theta\cos\theta - (1+2\sin\theta)\cos\theta}.$$

Plugging in $\theta = 0$ we obtain

$$\frac{dy}{dx} = \frac{1}{2}.$$

(b) Find an integral expression equal to the length of the inner loop of the curve. Do not evaluate the integral.

It's easiest to use the polar arclength formula, obtaining

$$L = \int_\alpha^\beta \sqrt{[f'(\theta)]^2 + [f(\theta]^2}\, d\theta = \int_{7\pi/6}^{11\pi/6} \sqrt{(2\cos\theta)^2 + (1+2\sin\theta)^2}\, d\theta.$$

We obtain the limits of integration by setting $r = 0$ and solving, which gives

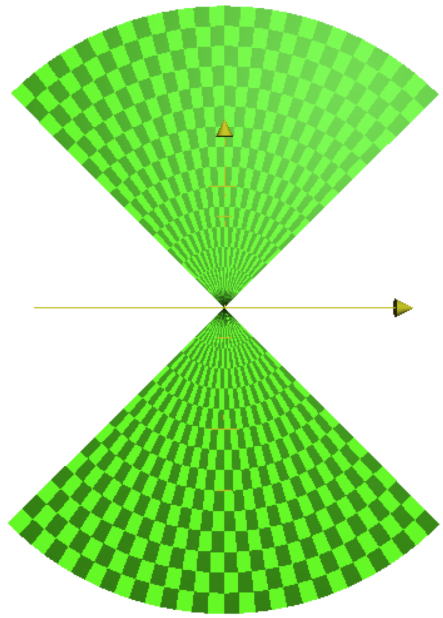
$$1 + 2\sin\theta = 0 \quad \Rightarrow \quad \sin\theta = -1/2 \quad \Rightarrow \quad \theta = 7\pi/6, 11\pi/6.$$

We use the picture (and the labels from part (a)) to determine the order of the limits.

3. (a) Sketch the region of the polar plane described by the following inequalities:

$$-1 \le r \le 1, \qquad \pi/4 \le \theta \le 3\pi/4.$$

(b) Find Cartesian coordinates for the point with polar coordinates $(2\sqrt{2}, 3\pi/4)$.



We use the standard equations:

$$x = r\cos\theta = 2\sqrt{2}\cos(3\pi/4) = -2 \qquad y = r\sin\theta = 2\sqrt{2}\sin(3\pi/4) = 2.$$

(c) Find two more sets of polar coordinates describing the point in part (b), at least one of which has $r < 0$.

We can subtract 2π from the angle without changing anything else, obtaining $(2\sqrt{2}, -5\pi/4)$, or we may subtract π from the angle and change the sign of r, obtaining $(-2\sqrt{2}, -\pi/4)$.

(d) Find the area inside the curve $r^2 = 8\cos(2\theta)$ and outside the curve $r = 2$.

We obtain the points of intersection by setting $r = r$, which gives

$$\sqrt{8\cos(2\theta)} = 2 \quad \Rightarrow \quad 8\cos(2\theta) = 4 \quad \Rightarrow \quad \cos(2\theta) = \frac{1}{2} \quad \Rightarrow \quad \theta.$$

Thus $2\theta = \pi/3$, so $\theta = \pi/6$ is one of the angles at which an intersection occurs. For simplicity, we will integrate from $\theta = 0$ to $\theta = \pi/6$ and quadruple the answer to get the total area. Thus the area is

$$A = 4\left(\frac{1}{2}\int_0^{\pi/6} (8\cos(2\theta) - 4)\, d\theta\right) = (8\sin(2\theta) - 8\theta)|_{\theta=0}^{\pi/6} = 4\sqrt{3} - \frac{16\pi}{3}.$$

4. Show that the curve $c(t) = (e^{-t}\cos t,\ e^{-t}\sin t)$ has finite length for $0 \le t < \infty$.

Here's a clever way to do it. Since the parametrization is of the form $c(t) = (f(t)\cos t,\ f(t)\sin t)$, there is a corresponding polar equation $r = f(t) = e^{-t}$, and we may apply the polar arc length formula to it, instead of using the parametric form for the given parametrization; the answer will be the same. (Note that t plays the role of θ, so the limits of integration are the same.) Thus we have

$$L = \int_0^\infty \sqrt{(e^{-t})^2 + (-e^{-t})^2}\, dt = \sqrt{2}\int_0^\infty e^{-t} = \sqrt{2}.$$

5A. Use the series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

to find the Taylor series (centered at zero) for the indefinite integral $\displaystyle\int \frac{x^2}{1+x}\, dx$. Also find the values of x for which the resulting series converges.

We first write

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - \cdots.$$

Now we multiply through by x^2, obtaining

$$\frac{x^2}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^{n+2} = x^2 - x^3 + x^4 - x^5 + x^6 - \cdots.$$

Now we integrate to get

$$\int \frac{x^2}{1+x}\, dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+3}}{n+3} = \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \cdots.$$

To find where this series converges, we note that the original series for $\frac{1}{1-x}$ converges for $-1 < x < 1$ (if you don't remember this, you can verify it with a ratio test). Because the substitution, the multiplication, and the integration do not change the radius of convergence, the new series also converges for $-1 < x < 1$.

When we check the endpoints, we note that $x = 1$ gives the series

$$\frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots,$$

which converges by the alternating series test, while when $x = -1$ we have

$$-\frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \cdots,$$

which is the negative of the harmonic series (minus the first two terms), and thus diverges.

5B. Consider the line segment joining the origin to the point (a, b). We have two ways of finding the length of this segment: the distance formula (i.e., the Pythagorean theorem) and the arc length integral. Show that both methods give the same answer.

The Pythagorean theorem implies that the length of the segment is $L = \sqrt{a^2 + b^2}$. To use an arclength integral, we need to parametrize the segment. Using the two points $(0, 0)$ and (a, b), we find that the equation for the line is $y = \frac{b}{a}x$, and thus has parametrization $c(t) = (t, \frac{b}{a}t)$ for $0 \leq t \leq a$. The formula then gives

$$L = \int_0^a \sqrt{1 + \frac{b^2}{a^2}}\, dt = a\sqrt{1 + \frac{b^2}{a^2}} = \sqrt{a^2 + b^2}.$$