BOUNDING SURFACE ACTIONS ON HYPERBOLIC SPACES

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Abstract. We give a diameter bound for fundamental domains for isometric actions of the fundamental group of a closed orientable hyperbolic surface on a δ-hyperbolic space, where the bound depends on the hyperbolicity constant δ, the genus of the surface, and the injectivity radius of the action, which we assume to be strictly positive.

1. Introduction

If S is a closed hyperbolic surface with genus g and injectivity radius bounded below by $\epsilon > 0$, then the diameter of S is bounded above by some constant D, depending only on g and ϵ. One way to prove this is as follows (cf. [Bon86, Lemma 1.10]). The total area of S is bounded in terms of the genus g and the curvature $\kappa = -1$. Now join any pair of points by a distance-realising path of length D. The injectivity radius assumption impies that such a path has an embedded product neighborhood with radius $\epsilon/3$, the area of which (by the negative curvature assumption) is at least $D\epsilon/3$. It follows that D is constrained by the global area bound.

This result can be rephrased so as to provide a bound for the diameter of a fundamental domain for the corresponding action of $\pi_1(S)$ on $\mathbf{H}^2$. In this context, we seek a generalization to actions of $\pi_1(S)$ on δ-hyperbolic spaces. In place of area bounds, which are no longer available, we exploit the fact that a large diameter requires the existence of short loops in the quotient.

Suppose Γ is a spine on S; i.e., Γ is a finite connected graph embedded in S with single disk complement. We let $\widetilde{\Gamma}$ denote the lift of Γ to the universal cover $\widetilde{S}$ of S. Suppose $\gamma\colon \widetilde{\Gamma} \to X$ is an embedding into a length-space X that is equivariant with respect to some isometric action of $\pi_1(S)$ on X. We obtain a pseudometric d_γ on Γ by declaring

$$d_\gamma(p,q) = \inf\{d_X(P,Q) \mid P \in \gamma(\tilde{p}),\, Q \in \gamma(\tilde{q})\}.$$

This will be a true metric when the injectivity radius of the action is strictly positive; i.e., when

$$\operatorname{inj}(X, \pi_1(S)) = \inf_{1 \neq h \in \pi_1(S)} \left(\lim_{n\to\infty} \frac{d_X(x, h^n x)}{n} \right) \geq \epsilon > 0.$$

From this metric we obtain a notion of diameter, given by

$$\operatorname{diam}_\gamma(\Gamma) = \max\{d_\gamma(a,b) \mid a, b \in \Gamma\}.$$

Our main result is the following.

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Theorem. *There is a constant* $D = D(\delta, g, \epsilon)$ *with the following property: Suppose* S *is a closed, orientable surface of genus* g*, and suppose* $\pi_1(S)$ *acts isometrically on a* δ*-hyperbolic space* X *with* $\operatorname{inj}(X, \pi_1(S)) \geq \epsilon > 0$*. Then there is a spine* Γ *embedded on* S *and a* $\pi_1(S)$*-equivariant embedding* $\gamma\colon \widetilde{\Gamma} \to X$ *so that* $\operatorname{diam}_\gamma(\Gamma) \leq D$*.*

Remark 1.1. We originally proved (a slightly less general version of) this theorem in the appendix to [Bar04]. A slightly different formulation of this theorem says that given a fixed generating set for $\pi_1(S)$, there is some point $x \in X$ that is translated a bounded distance by each element in that generating set. See [Bow07a] for more details. In [Bow07b] Bowditch shows how to adapt the proof here to apply also to surfaces with boundary, as well as to nonorientable surfaces.

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2. Preliminaries

We first fix some notation that will hold for the rest of the paper. We let S denote a closed orientable surface of genus g, and we assume we have a fixed isometric action of $\pi_1(S)$ on a complete geodesic path-metric space (X, d_X) with δ-thin triangles. The injectivity radius of this action is assumed to be bounded below by $\epsilon > 0$.

Let $\widetilde{S}$ denote the universal cover of S, and for any subset $Z \subset S$, we let $\widetilde{Z}$ denote the full lift of Z in $\widetilde{S}$. A *carrier graph* for the action of $\pi_1(S)$ on X is a pair (Γ, γ), where Γ is an embedded graph $\Gamma \subseteq S$ carrying all of $\pi_1(S)$ and $\gamma\colon \widetilde{\Gamma} \to X$ is a $\pi_1(S)$-equivariant map sending edges of Γ to rectifiable paths in X. Given a carrier graph (Γ, γ), we obtain a metric d_γ as described above, and we let $\ell_\gamma(\Gamma)$ denote the sum of the lengths of the edges of Γ measured in this way. We abuse notation by using d_γ, ℓ_γ, and $\operatorname{diam}_\gamma$ also for arbitrary subsets of S.

We say that (Γ, γ) is *minimal* if $\ell_\gamma(\Gamma) = \inf\{\ell_\sigma(\Sigma)\}$, where the infimum is taken over all carrier graphs (Σ, σ) for the given action. Note that if (Γ, γ) is a minimal carrier graph, then we may assume that Γ is a spine on S. For convenience we will assume all vertices of Γ are at least trivalent.

Remark 2.1. A priori this infimum may not be achieved, in which case one needs to work with *almost minimal* carrier graphs, and then let the error term go to zero. For convenience of exposition, we will assume that a minimal carrier graph exists for the given action.

3. Proof

We say a set $Z \subseteq X$ has *controlled diameter* if it is contained in the $\pi_1(S)$-orbit of a bounded number of sets, each with bounded diameter, where all bounds depend on δ, g, and ϵ. We verify the theorem by showing that $\gamma(\widetilde{\Gamma})$ has controlled diameter.

Fix a minimal spine (Γ, γ) and let Δ denote the closure in $\widetilde{S}$ of a single lift of the complement $S - \Gamma$. A simple Euler characteristic argument shows that the number of edges of any spine is bounded above by $6g - 3$, so it follows from the minimality assumption that $\gamma(\partial\Delta)$

is a geodesic n-gon in X, where $n \leq 12g - 6$. Note that the action of $\pi_1(S)$ on X identifies sides of $\gamma(\partial\Delta)$ in pairs so that the quotient is Γ.

Let $\Gamma^{(0)}$ denote the vertices of Γ, and define B_0 to be the $n\delta$-neighborhood of $\gamma(\widetilde{\Gamma}^{(0)})$ in X. Note that B_0 has controlled diameter, as the number of vertices of Γ is bounded in terms of g. Suppose there is some point $\tilde{s} \in \partial\Delta$ with $\gamma(\tilde{s}) \notin B_0$. Because $\gamma(\partial\Delta)$ is a geodesic n-gon in a δ-hyperbolic space, there is a point $\tilde{s}'$ on $\partial\Delta$ with $d_X(\gamma(\tilde{s}), \gamma(\tilde{s}')) \leq n\delta$. In fact, we may join $\tilde{s}$ to $\tilde{s}'$ by a path $\tilde{\sigma}$ properly embedded in Δ and extend γ to $\tilde{\sigma}$ so that $\ell_\gamma(\tilde{\sigma}) \leq n\delta$.

Claim. *The two points $\tilde{s}$ and $\tilde{s}'$ project into S to the same edge e of Γ. Moreover, the γ-length of that portion of e bounded by the projections $s, s' \in \Gamma$ has length no more than $n\delta$.*

Proof. Suppose first that s and s' lie on distinct edges e and e' of Γ, and let σ be the projection of $\tilde{\sigma}$ into S. We will obtain a contradiction by constructing a carrier graph (Γ', γ') with $\ell_{\gamma'}(\Gamma') < \ell_\gamma(\Gamma)$. We construct Γ' from Γ by removing a component of $e - s$ and inserting the path σ, and then γ' is simply the appropriate restriction of γ (recall that we had extended γ to $\tilde{\sigma}$).

Because the complement of Γ is a disk, the complement of $\Gamma \cup \sigma$ is two disks. It follows that Γ' has a single disk complement and thus carries $\pi_1(S)$. Because $\gamma(\tilde{s})$ lies outside B_0, the segment of Γ deleted in forming Γ' has γ-length greater than $n\delta$, while σ has γ'-length less than $n\delta$. This contradicts the assumed minimality of (Γ, γ). We deduce that $e = e'$.

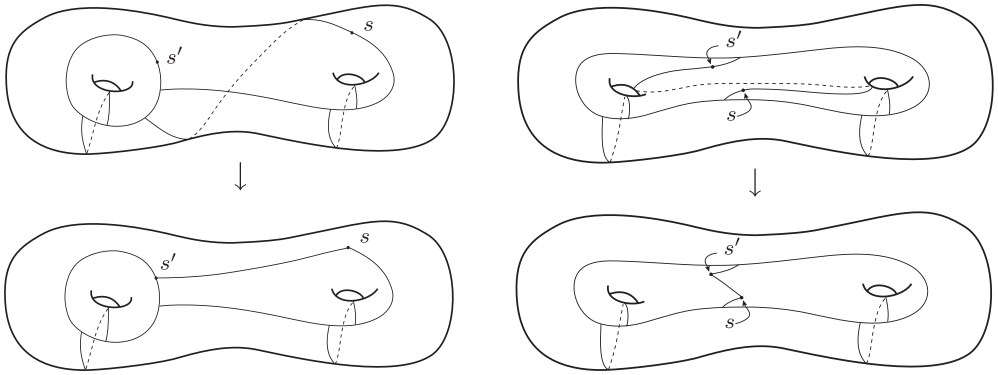


FIGURE 1. Two ways to shorten a carrier graph (the lengths suggested by the pictures are γ-lengths).

Now suppose the γ-length of the portion of e between s and s' is greater than $n\delta$. We will again arrive at a contradiction by constructing a shorter carrier graph. This time we construct Γ' by deleting that portion of e between s and s' and inserting σ, and define γ' as before. The result follows. □

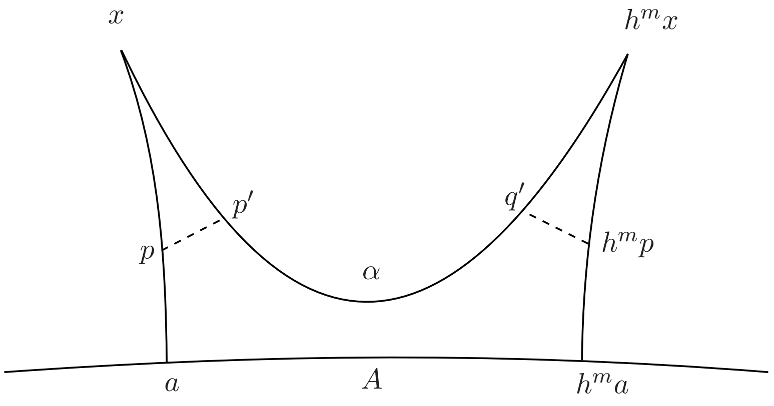
It follows from this claim that σ and that portion of e between s and s' together form a simple loop β_1 on S. This loop is homotopically essential, as it does not lift to a loop in $\widetilde{S}$. Also note that the claim implies that $\ell_\gamma(\beta_1) \leq 2n\delta$. Thus from the existence of a point on $\partial\Delta$ mapping outside B_0, we construct an essential closed curve on S with γ-length no more than $2n\delta$.

The plan now is to enlarge B_0 to B_1, which will include a neighborhood of the $\pi_1(S)$-orbit of $\gamma(\tilde{\beta}_1)$. We then proceed as before, showing that the existence of a point of $\gamma(\partial\Delta)$ outside B_1 provides another simple closed curve on S with short γ-length. Moreover, this curve will be disjoint from and not freely-homotopic to the previous one. For all of this we need the following lemma.

Lemma 3.1. *There is a constant K, depending only on δ, n, and ϵ, so that if x and x' are two points in X with the property that $d_X(x, hx) \leq 2n\delta$ and $d_X(x', hx') \leq 2n\delta$ for some $h \in \pi_1(S)$, then the Hausdorff distance between $\{h^n x\}$ and $\{h^n x'\}$ is bounded above by K.*

Proof. Here is a sketch of the proof. Because of the injectivity radius lower bound, every element of $\pi_1(S)$ has a quasi-axis in X; i.e., for each $h \in \pi_1(S)$ there is a quasigeodesic A_h moved by the element a bounded Hausdorff distance from itself, where the bounds all depend only on δ. We claim that the farther a point is from this quasiaxis, the greater the distance it is moved by h. Thus an upper bound on the distance a point is moved by h gives an upper bound of the distance from the point to the quasiaxis, where the bound depends on δ as well as the stable translation length of h, which in turn is bounded below by ϵ. In particular, two such points are boundedly Hausdorff close to the quasiaxis, and hence to one another.

To prove the claim, pick $x \in X$, and suppose $d_X(x, A_h) = d_X(x, a) = L$ for some point a on the quasiaxis A_h. Fix $m = \lceil 2\delta/\epsilon \rceil$, so that $d_X(a, h^m x) \geq 2\delta$. Let α denote a geodesic segment joining x to $h^m x$. For sufficiently large L, we may choose p on a geodesic segment joining x to a so that $\delta < d_X(p, a) < 2\delta$. Now suppose A_h is actually geodesic. Then we also have $\delta < d_X(p, A_h) < 2\delta$, so by the thin triangles condition applied to the quadrilateral $(a, x, h^m x, h^m a)$, there is some point $p' \in \alpha$ so that $d_X(p, p') \leq \delta$. By the triangle inequality, we have $d_X(x, p') \geq L - 3\delta$. We similarly find q' with$d_X(h^m x, q') \geq L - 3\delta$. By choice of m, the portion of α joining x to p' does not intersect the portion joining hx to q'. It follows that $d_X(x, h^m x) \geq 2L - 6\delta$. Dividing by $2\delta/\epsilon$, we find that $d_X(x, hx) \geq \frac{L\epsilon}{\delta} - 3\epsilon$, so that h-translation distance is an unboundedly increasing function of L, as claimed. When A_h is not geodesic, we apply the argument to a geodesic segment $\overline{A}$ joining a to $h^m a$ and use the fact that the corresponding portion of A_h lies within a bounded Hausdorff neighborhood of $\overline{A}$. □



Using this constant K, we define B_1 to be the union of B_0 with the $\max\{K, 4n\delta\}$-neighborhood of $\gamma(\tilde{\beta}_1)$. Note that B_1 has controlled diameter. This is because the γ-length of β_1 is bounded appropriately, as is the diameter of the neighborhood taken.

For general $j \geq 1$, suppose B_{j-1} has been defined, and suppose $\gamma(\partial\Delta) \nsubseteq B_{j-1}$. Then we may use a point on $\partial\Delta$ mapping outside B_{j-1} to construct a simple closed curve β_j on S with γ-length (for γ extended as before) no more than $2n\delta$. We then inductively define B_j to be the union of B_{j-1} with the $\max\{K, 4n\delta\}$-neighborhood of $\gamma(\tilde{\beta}_j)$. Note that B_j has controlled diameter.

We claim now that $\beta_j \cap \beta_i = \emptyset$ for $i \neq j$, and that β_j and β_i are not freely homotopic. For the first part, if the two curves intersected in some point p, then because each of β_j and β_i has γ-length no more than $2n\delta$, one could construct a path from any point on β_j to any point on β_i, through p, with total γ-length no more than $4n\delta$. This contradicts the fact at least one point on $\gamma(\tilde{\beta}_j)$ lies outside the $4n\delta$-neighborhood of $\gamma(\tilde{\beta}_i)$ for $i < j$.

For the second part, because at least one point of $\gamma(\tilde{\beta}_j)$ lies outside the K-neighborhood of all previous $\gamma(\tilde{\beta}_i)$, the lemma above implies that β_j cannot be freely homotopic to any β_i for $i < j$.

The result follows when we note that the maximum number of non-parallel disjoint simple closed curves β_j one may find on S is $2g - 1$, so that $\gamma(\partial\Delta)$ must be contained in B_k for some $k \leq 2g - 1$. But the $\pi_1(S)$-orbit of $\gamma(\partial\Delta)$ is all of $\gamma(\widetilde{\Gamma})$. Thus $\gamma(\widetilde{\Gamma})$ is contained in the set B_k, which has controlled diameter. The result follows.

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