

Extensions for finite groups of Lie type II: Filtering the truncated induction functor

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Dedicated to James E. Humphreys on the occasion of his 65th birthday

ABSTRACT. In [BNP5] the authors relate the extensions between two simple modules for a finite group of Lie type $G_\sigma(\mathbb{F}_q)$ (where $q = p^r$) to certain extensions for the corresponding reductive group and its Frobenius kernels. Several of these results require the characteristic p of the underlying field to be sufficiently large ($p \geq 3(h-1)$, with h being the Coxeter number of the root system). In this paper we will generalize these results to all primes p assuming instead lower bounds on the prime powers p^r (approximately of the order of h^2).

1. Introduction

1.1. Let G be a connected reductive algebraic group scheme defined over \mathbb{F}_p and let $F : G \rightarrow G$ be the Frobenius map. Let G_r be the r -th Frobenius kernel which is the scheme theoretic kernel of F^r (F composed with itself r times) and let $G(\mathbb{F}_q)$ be the fixed points under F^r . We will assume that k is an algebraically closed field of characteristic $p > 0$. The finite groups $G(\mathbb{F}_q)$ are called the finite Chevalley groups. There has been much effort in the last thirty years aimed at understanding the interrelationships between the representation theory of these three algebraic objects. For a comprehensive treatment of this subject we refer the reader to Humphreys' book [Hum2].

In a series of papers [BNP1, BNP2, BNP3, BNP5] the authors investigated the deep connections between the cohomology theories of G , G_r and $G(\mathbb{F}_q)$. The philosophy behind our approach involved using certain truncated categories of rational G -modules which approximate the categories of G_r and $G(\mathbb{F}_q)$ -modules. These truncated categories are highest weight categories and contain enough projective modules so one can directly compare these categories to the categories of G_r and

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$G(\mathbb{F}_q)$ -modules through the use of Grothendieck spectral sequences. In the construction of these spectral sequences, we study certain families of finite dimensional submodules of the infinite dimensional induced module $\text{ind}_{G(\mathbb{F}_q)}^G(N)$ where $N \in \text{mod}(kG(\mathbb{F}_q))$. These modules can be described as the maximal submodules of $\text{ind}_{G(\mathbb{F}_q)}^G(N)$ whose highest weights are contained in specified finite saturated sets of weights. An important example occurs when we let N be the trivial module. For large primes ($p \geq 3(h-1)$) and truncation at approximately twice the Steinberg weight, it was shown that this module (when $N \cong k$) is completely reducible ([BNP1, BNP5]). This fact led to explicit formulas describing extensions of simple modules over $G(\mathbb{F}_q)$ via extensions of modules for G [BNP2, BNP3, BNP5]. As an application, we were able to use our formulation to answer many of the questions posed in Humphreys' 1985 article on self-extensions [Hum1].

In this paper, we will use the same setup as in [BNP5] and consider the more general family of finite groups of Lie type. We denote these groups by $G_\sigma(\mathbb{F}_q)$ where σ is the corresponding automorphism of G . For small primes these truncated induction functors are no longer semisimple. Our goal is to study the resulting modules for small primes p , but large prime powers p^r .

The precise definitions and some basic properties of these truncated categories and associated functors are given in Section 2. Then some useful cohomology facts will be noted in Section 3. Section 4 is devoted to demonstrating that under suitable conditions on p^r these truncated induced modules admit a filtration with sections of the form $H^0(-w_0\sigma\mu) \otimes H^0(\mu)^{(r)}$ (Theorem 4.7).

In Section 5, we apply this filtration to make some cohomological computations. For example, the existence of this filtration allows us to show that for $r \geq 2$ and sufficiently large $q = p^r$ the finite group $G_\sigma(\mathbb{F}_q)$ does not allow self-extensions between simple modules (Theorem 5.4). Tiep and Zalesskii [TZ, Prop. 1.4] have shown that the existence of self-extensions are an important factor in the ability to lift irreducible representations from characteristic p to characteristic zero.

Finally, in Section 5.6, it is shown that for all primes but $r \geq 3$ and q sufficiently large the group of extensions between two simple $G_\sigma(\mathbb{F}_q)$ -modules is isomorphic to the G -extensions between a suitable pair of q -restricted simple G -modules. Roughly speaking one can say that $\text{Ext}_{G_\sigma(\mathbb{F}_q)}^1$ for pairs of simple $G_\sigma(\mathbb{F}_q)$ -modules mirrors the theory of Ext_G^1 between q -restricted simple modules, provided that $r \geq 3$ and q is at least of the order of the Coxeter number squared. No restriction on the prime is necessary.

1.2. Notation. Let G be a connected simply connected almost simple algebraic group defined and split over the finite field \mathbb{F}_p with p elements and k be the algebraic closure of \mathbb{F}_p . We will also consider G as an algebraic group scheme over \mathbb{F}_p .

Let Φ be a root system associated to the pair (G, T) where T is a maximal split torus. Moreover, let Φ^+ (resp. Φ^-) be the positive (resp. negative) roots and Δ be a base consisting of simple roots.

Let $X(T)$ be the integral weight lattice obtained from Φ contained in the Euclidean space \mathbb{E} with the inner product denoted by $\langle \cdot, \cdot \rangle$. The set $X(T)$ has a partial ordering given by $\lambda \geq \mu$ if and only if $\lambda - \mu \in \sum_{\alpha \in \Delta} \mathbb{N}\alpha$ for $\lambda, \mu \in X(T)$. The set of dominant integral weights is denoted by $X(T)_+$ and the set of p^r -restricted weights by $X_r(T)$.

Let W be the Weyl group. The group W acts on $X(T)$ via the “dot action” given by $w \cdot \lambda = w(\lambda + \rho) - \rho$ where ρ is the half sum of the positive roots. Let $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ be the coroot corresponding to $\alpha \in \Phi$. The longest element in W is w_0 and the Coxeter number for Φ is $h = \langle \rho, \alpha_0^\vee \rangle + 1$ where α_0 is the maximal short root.

Let B be a Borel subgroup containing T corresponding to the negative roots. For $\lambda \in X(T)_+$ set $H^0(\lambda) = \text{ind}_B^G \lambda$. The simple G -module corresponding to λ is denoted by $L(\lambda)$ and the Weyl module is $V(\lambda)$. The injective hull of $L(\lambda)$ as a G -module will be denoted $I(\lambda)$. For more details about the definitions and properties of these objects we refer the reader to [Jan1].

Let $F : G \rightarrow G$ be the Frobenius map and F^r the composition of the Frobenius map with itself r -times. Now suppose that σ is an automorphism of the Dynkin diagram of Φ . The automorphism σ can be extended to the weight lattice $X(T)$ and under this extension σ permutes the fundamental weights and preserves the inner product $\langle \cdot, \cdot \rangle$ as well as the partial order on $X(T)$. Moreover, $\sigma(\alpha_0) = \alpha_0$. The graph automorphism σ also induces an automorphism on G which will also be denoted by σ . The automorphism σ commutes with F and is compatible with the action of σ on $X(T)$. Set $G_\sigma(\mathbb{F}_q)$ as the group of fixed points of $F^r \circ \sigma = \sigma \circ F^r$, where $q = p^r$. The groups $G_\sigma(\mathbb{F}_q)$ can be either (i) untwisted (Chevalley) groups, (ii) Steinberg groups, or (iii) Suzuki-Ree groups. For more information about these groups see [Car] [GLS]. For simplicity we will exclude the Suzuki-Ree groups from our discussion. With some exceptions for the Ree groups of type F_4 , the extensions for these groups are known due to [Sin1, Sin2, Sin3].

Throughout this paper, for $\nu \in X(T)$, set $\hat{\nu} = -w_0\sigma\nu$.

2. Induction and Truncation.

2.1. Induction. For a finite dimensional $G_\sigma(\mathbb{F}_q)$ -module M and a finite dimensional G_r -module N , we define

$$\mathcal{G}(M) = \text{ind}_{G_\sigma(\mathbb{F}_q)}^G(M) \text{ and } \mathcal{H}(N) = \text{ind}_{G_r}^G(N).$$

In particular for the trivial module k we set $\mathcal{G}(k) = \text{ind}_{G_\sigma(\mathbb{F}_q)}^G(k)$ and $\mathcal{H}(k) = \text{ind}_{G_r}^G(k)$. If M and N are G -modules, the tensor identity implies

$$\mathcal{G}(M) = M \otimes \text{ind}_{G_\sigma(\mathbb{F}_q)}^G(k) \text{ and } \mathcal{H}(N) = N \otimes \text{ind}_{G_r}^G(k).$$

Our first result shows that $\mathcal{G}(k)$ (resp. $\mathcal{H}(k)$) is injective upon restriction to G_r (resp. $G_\sigma(\mathbb{F}_q)$).

PROPOSITION .

- (i) $\mathcal{H}(k)$ is injective as a $G_\sigma(\mathbb{F}_q)$ -module.
- (ii) $\mathcal{G}(k)$ is injective as a G_r -module.

PROOF. (i) It is well-known that

$$\mathcal{H}(k) \cong k[G/G_r] \cong k[G]^{(r)} \cong \bigoplus_{\nu \in X(T)_+} (I(\nu)^{(r)})^{\dim L(\nu)}$$

as a G -module where $I(\nu)$ is the injective hull of the simple module $L(\nu)$. Moreover, $I(\nu) \cong I(\nu)^{(r)}$ as a $G_\sigma(\mathbb{F}_q)$ -module and is injective since $G/G_\sigma(\mathbb{F}_q)$ is affine.

(ii) As a $G_\sigma(\mathbb{F}_q)$ -module, the Steinberg module St_r is both projective and injective. Furthermore, $\text{St}_r \cong \text{St}_r^{(r)}$ as $G_\sigma(\mathbb{F}_q)$ -modules. The functor \mathcal{G} sends injective

$G_\sigma(\mathbb{F}_q)$ -modules to injective G -modules. Therefore, from the tensor identity we obtain the following sequence of isomorphisms of injective G -modules:

$$\mathcal{G}(\mathrm{St}_r) \cong \mathcal{G}(\mathrm{St}_r^{(r)}) \cong \mathrm{St}_r^{(r)} \otimes \mathcal{G}(k).$$

Restriction from G -modules to G_r -modules sends injectives to injectives (because G/G_r is affine) and the r -th Frobenius twist of St_r , viewed as a module for G_r , is a direct sum of trivial modules. Hence, as G_r -modules, $\mathcal{G}(\mathrm{St}_r) \cong \mathcal{G}(k)^{\dim \mathrm{St}_r}$ and the assertion follows. \square

2.2. Saturated sets of weights. For any finite set of weights $\pi \subseteq X(T)_+$ we define $\mathcal{G}_\pi(M)$ (resp. $\mathcal{H}_\pi(N)$) to be the maximal G -submodule of $\mathcal{G}(M)$ (resp. $\mathcal{H}(N)$) having composition factors with highest weights in π . The following three sets of weights will play an important role in the upcoming results:

$$\begin{aligned} \Gamma &= \{\lambda \in X(T)_+ \mid \langle \lambda, \alpha_0^\vee \rangle < 2(h-1)\} \\ \Omega &= \{\lambda \in X(T)_+ \mid \langle \lambda, \alpha_0^\vee \rangle < 2p^r(h-1)\} \\ \Lambda &= \{\lambda \in X(T)_+ \mid \langle \lambda, \alpha_0^\vee \rangle < 3(p^r-1)(h-1)\}. \end{aligned}$$

Our goal is to understand the structure of the module $\mathcal{G}_\Omega(k)$ for arbitrary (especially small) primes p and large p^r . We begin by constructing an ascending chain of submodules for $\mathcal{G}_\Omega(k)$. Fix an order $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ of the elements in Γ such that $i < j$ whenever $(p^r - w_0\sigma)\lambda_i < (p^r - w_0\sigma)\lambda_j$. Notice that $i < j$ whenever $\lambda_i < \lambda_j$. Clearly $\lambda_1 = 0$. Then we define subsets of Γ as follows, for $i = 1, 2, \dots, n$, set

$$\Gamma_i = \{\lambda_j \in \Gamma \mid j \leq i\} \text{ and } \Gamma_0 = \emptyset.$$

Let $\gamma_0, \gamma_1 \in \Gamma$. Then $\langle \gamma_1, \alpha_0^\vee \rangle \leq 2(h-1) - 1$ and $\langle \gamma_0 + p^r\gamma_1, \alpha_0^\vee \rangle < 2p^r(h-1) - p^r + 2(h-1)$. If $p^r \geq 2(h-1)$, then $\gamma_0 + p^r\gamma_1 \in \Omega$. For the remainder of this section we assume that $p^r \geq 2(h-1)$. We define subsets of Ω :

$$\Omega_i = \{\gamma \in X(T)_+ \mid \gamma \leq (p^r - w_0\sigma)\nu \text{ for some } \nu \in \Gamma_i\}.$$

The subsets Γ_i and Ω_i together with their W -conjugates are saturated, $\Gamma_0 = \Omega_0 = \emptyset$, and $\Gamma_1 = \Omega_1 = \{0\}$. We have the following series of inclusions

$$(2.2.1) \quad k = \mathcal{G}_{\Omega_0}(k) \subseteq \mathcal{G}_{\Omega_1}(k) \subseteq \mathcal{G}_{\Omega_2}(k) \subseteq \dots \subseteq \mathcal{G}_{\Omega_n}(k) \subseteq \mathcal{G}_\Omega(k).$$

Later we will show that $\mathcal{G}_{\Omega_n}(k) = \mathcal{G}_\Omega(k)$ for sufficiently large p^r .

REMARK . In [BNP2, BNP5] the notation $\mathcal{G}(k)$ is used for a truncated submodule of $\mathrm{ind}_{G_\sigma(\mathbb{F}_q)}^G(k)$. Here $\mathcal{G}(k)$ will always denote the infinite-dimensional module $\mathrm{ind}_{G_\sigma(\mathbb{F}_q)}^G(k)$ itself. Any finite-dimensional truncated submodule will be denoted by $\mathcal{G}_\pi(k)$ with π being the corresponding finite set of weights.

2.3. Injectives and projectives in the truncated categories. Let π be a finite set of dominant weights such that π together with its W -conjugates is saturated. Let $\mathrm{Mod}(\pi)$ denote the full subcategory of $\mathrm{Mod}(G)$ with objects having composition factors whose highest weights lie in π . Such a truncated category has both injective and projective modules. For a weight $\lambda \in \pi$, we denote the injective hull and the projective cover of the simple module $L(\lambda)$ by $I_\pi(\lambda)$ and $P_\pi(\lambda)$, respectively. The module $I_\pi(\lambda)$ can be described as the maximal G -submodule of the injective hull $I(\lambda)$ of $L(\lambda)$ in $\mathrm{Mod}(G)$ whose composition factors have weights in π . In particular, the module $I_\pi(\lambda)$ is finite dimensional and has a good filtration.

Moreover, for any $\gamma \in \pi$, the multiplicity of the factor $H^0(\gamma)$ in a good filtration of $I_\pi(\lambda)$, denoted by $[I_\pi(\lambda) : H^0(\gamma)]_G$, equals the multiplicity of the simple module $L(\lambda)$ as a composition factor in $H^0(\gamma)$, denoted by $[H^0(\gamma) : L(\gamma)]_G$. For a general treatment of truncated categories we refer to [Don1, Don2] or [Jan1, II.A].

Next consider the automorphism σ on G and $G_\sigma(\mathbb{F}_q)$. One obtains for G -modules M and N

$$(2.3.1) \quad \text{Ext}_G^i(M, N) \cong \text{Ext}_G^i(M^{\sigma^{-1}}, N^{\sigma^{-1}}) \text{ for } i \geq 0.$$

The Frobenius morphism F^r is also an automorphism on $G_\sigma(\mathbb{F}_q)$ with $F^r \circ \sigma = \sigma \circ F^r$ being the identity. Hence

$$(2.3.2) \quad \text{Ext}_{G_\sigma(\mathbb{F}_q)}^i(M, N) \cong \text{Ext}_{G_\sigma(\mathbb{F}_q)}^i(M^{\sigma^{-1}}, N^{\sigma^{-1}}) \cong \text{Ext}_{G_\sigma(\mathbb{F}_q)}^i(M^{(r)}, N^{(r)}) \text{ for } i \geq 0.$$

Moreover, from [Jan2, 1.3] one concludes for $\lambda \in X(T)^+$ that

$$L(\lambda)^{\sigma^{-1}} \cong L(\sigma\lambda) \text{ as a } G\text{-module,}$$

and

$$L(\lambda)^{\sigma^{-1}} \cong L(\sigma\lambda) \cong L(\lambda)^{(r)} \text{ as a } G_\sigma(\mathbb{F}_q)\text{-module.}$$

It follows that

$$(2.3.3) \quad P_\pi(\lambda)^{\sigma^{-1}} \cong P_{\sigma(\pi)}(\sigma\lambda) \text{ and } [P_{\sigma(\pi)}(\sigma\lambda) : L(\sigma\gamma)]_G = [P_\pi(\lambda) : L(\gamma)]_G.$$

Also

$$(2.3.4) \quad P_\pi(\lambda)^{(r)} \cong P_\pi(\lambda)^{\sigma^{-1}} \cong P_{\sigma(\pi)}(\sigma\lambda) \text{ as a } G_\sigma(\mathbb{F}_q)\text{-module.}$$

Finally, note that the projective module $P_\pi(\lambda)$ is isomorphic to the dual module of $L_{-w_0(\pi)}(-w_0\lambda)$.

3. Cohomological Facts

In this section, we record several cohomological results which will be used a number of times later in the paper.

3.1. The following gives a condition under which homomorphisms over G_r may be identified with those over G .

PROPOSITION . *Assume $\lambda, \mu \in X_r(T)$ and M is a finite dimensional rational G -module such that all its weights ν satisfy $\langle \nu, \alpha_0^\vee \rangle < p^r$. Then*

$$\text{Hom}_{G_r}(L(\lambda), L(\mu) \otimes M) = \text{Hom}_G(L(\lambda), L(\mu) \otimes M).$$

PROOF. Without loss of generality (by dualizing if necessary), we may assume that $\langle \mu, \alpha_0^\vee \rangle \leq \langle \lambda, \alpha_0^\vee \rangle$. All G -composition factors of $\text{Hom}_{G_r}(L(\lambda), L(\mu) \otimes M)$ are G_r -trivial so must be of the form $L(\gamma)^{(r)}$. For such a factor, $\lambda + p^r\gamma$ is a weight of $L(\mu) \otimes M$ and hence

$$\langle \lambda + p^r\gamma, \alpha_0^\vee \rangle \leq \langle \mu + \nu, \alpha_0^\vee \rangle \leq \langle \lambda, \alpha_0^\vee \rangle + \langle \nu, \alpha_0^\vee \rangle$$

for a weight ν of M . Hence, $p^r\langle \gamma, \alpha_0^\vee \rangle \leq \langle \nu, \alpha_0^\vee \rangle < p^r$ and so we must have $\gamma = 0$. Therefore $\text{Hom}_{G_r}(L(\lambda), L(\mu) \otimes M)$ has a trivial G -structure and the claim follows since

$$\text{Hom}_G(L(\lambda), L(\mu) \otimes M) = \text{Hom}_{G/G_r}(k, \text{Hom}_{G_r}(L(\lambda), L(\mu) \otimes M)).$$

□

3.2. Ext¹ for modules with small highest weights. Here we give an upper bound on the size of the weights of a G -module M to insure that $H^1(G_r, M) \cong H^1(G, M)$.

LEMMA . Assume that the root system Φ of G is not of type A_1 . Let M be a finite dimensional rational G -module whose highest weights λ satisfy $\langle \lambda, \alpha_0^\vee \rangle < p^{r-1}(p-1)$.

- (a) If $p \neq 2$ or Φ is not of type C_n , then $H^1(G_r, M) \cong H^1(G, M)$.
- (b) If $p = 2$, Φ is of type C_n , and k is not a composition factor of M , then $H^1(G_r, M) \cong H^1(G, M)$.
- (c) If $p = 2$, Φ is of type C_n , and k is a composition factor of M , then $H^1(G_r, M) \cong H^1(G, M) \oplus (\oplus_l L(\omega_1))^{(r)}$, where $l \geq 0$ and ω_1 denotes the first fundamental weight of Φ .

PROOF. Consider the Lyndon-Hochschild-Serre (LHS) spectral sequence

$$E_2^{i,j} = H^i(G/G_r, H^j(G_r, M)) \Rightarrow H^{i+j}(G, M).$$

If M has a composition factor of the form $L(\gamma)^{(r)}$, the assumption that $\langle p^r \gamma, \alpha_0^\vee \rangle < p^r$ implies that $\gamma = 0$. Hence, $\text{Hom}_{G_r}(k, M) \cong \text{Hom}_G(k, M)$. Therefore,

$$E_2^{1,0} = H^1(G/G_r, \text{Hom}_{G_r}(k, M)) \cong H^1(G/G_r, k) \otimes \text{Hom}_G(k, M) = 0$$

and so

$$H^1(G, M) = E^1 \cong E_2^{0,1} = \text{Hom}_{G/G_r}(k, H^1(G_r, M)).$$

Thus the isomorphisms in parts (a) and (b) hold if the highest weight of $H^1(G_r, M)$ is zero. Since the weight ω_1 is not contained in the root lattice, part (c) follows if the highest weights of $H^1(G_r, M)$ are zero or ω_1 .

By induction on a composition series for M it now suffices to prove the assertion for a simple module $L(\lambda)$. If $\lambda = 0$, then it follows from [And1] that

$$H^1(G_r, k) = \begin{cases} L(\omega_1)^{(r)} & \text{if } p = 2 \text{ and } \Phi \text{ is of type } C_n \\ 0 & \text{else.} \end{cases}$$

For $\lambda \neq 0$ define the quotient Q via the short exact sequence

$$0 \rightarrow L(\lambda) \rightarrow H^0(\lambda) \rightarrow Q \rightarrow 0$$

and consider a portion of the associated long exact sequence

$$\begin{aligned} \text{Hom}_{G_r}(k, L(\lambda)) \hookrightarrow \text{Hom}_{G_r}(k, H^0(\lambda)) &\rightarrow \text{Hom}_{G_r}(k, Q) \\ &\rightarrow H^1(G_r, L(\lambda)) \rightarrow H^1(G_r, H^0(\lambda)). \end{aligned}$$

The size of λ forces all composition factors of Q and $H^0(\lambda)$ to be p^r -restricted. This implies that all Hom_{G_r} in the above sequence can be replaced by Hom_G giving

$$\begin{aligned} \text{Hom}_G(k, L(\lambda)) \hookrightarrow \text{Hom}_G(k, H^0(\lambda)) &\rightarrow \text{Hom}_G(k, Q) \\ &\rightarrow H^1(G_r, L(\lambda)) \rightarrow H^1(G_r, H^0(\lambda)). \end{aligned}$$

The first map is an isomorphism and $\text{Hom}_G(k, Q) \cong H^1(G, L(\lambda))$ by [Jan1, II.2.14]. It is therefore sufficient to show that $H^1(G_r, H^0(\lambda)) = 0$. It follows from [BNP4, 3.2] that $H^1(G_r, H^0(\lambda)) = 0$ unless $\lambda = p^r \nu - p^i \alpha$ with $\nu \in X(T)$, $\alpha \in \Delta$, and $0 \leq i < r$. Since λ is dominant and not zero, we have $\langle \nu, \alpha_0^\vee \rangle \geq 1$. For all root systems other than A_1 and C_2 , one has $\langle \alpha, \alpha_0^\vee \rangle \leq 1$. Hence, $\langle \lambda, \alpha_0^\vee \rangle < p^r - p^{r-1}$ implies $H^1(G_r, H^0(\lambda)) = 0$. If Φ is of type C_2 , the above argument fails in the

case $\lambda = p^r \nu - p^{r-1} \alpha$ with α being the long simple root. However this case is not of interest because λ being dominant forces $\langle \nu, \alpha_0^\vee \rangle \geq 2$ and $\langle p^r \nu - p^{r-1} \alpha, \alpha_0^\vee \rangle \geq 2p^r - 2p^{r-1} > p^{r-1}(p-1)$. \square

REMARK . Direct computation shows that the Proposition also holds for type A_1 and $p = 2$. For type A_1 and odd primes, one obtains $H^1(G_r, M) \cong H^1(G, M)$ for all finite dimensional G -modules with highest weights λ satisfying $\langle \lambda, \alpha_0^\vee \rangle < p^{r-1}(p-2)$.

3.3. Vanishing of certain Ext_G^1 -groups. It is well-known that a finite-dimensional G -module admits a good filtration if and only if $\text{Ext}_G^1(V(\mu), M) = 0$ for all $\mu \in X(T)_+$. Our goal in Section 4 is to show that certain modules have filtrations with factors of the form $H^0(\widehat{\mu}_0) \otimes H^0(\mu_1)^{(r)}$ where $\mu_0, \mu_1 \in \Gamma$. In order to establish these results, we need the following proposition. Here we set $\widehat{\Gamma}_i = -w_0 \sigma(\Gamma_i)$.

PROPOSITION . *Let $p^{r-1}(p-1) > 4h-6$ and $\gamma_0, \gamma_1, \mu_0, \mu_1 \in \Gamma_i$. Assume that the root system Φ of G is not of type A_1 . If $p = 2$ and Φ is of type C_n , then we assume in addition that $\gamma_0 - \gamma_1$ and $\mu_0 - \mu_1$ are contained in the root lattice. Then the following hold:*

- (i) $\text{Ext}_G^1(V(\widehat{\mu}_0) \otimes V(\mu_1)^{(r)}, H^0(\widehat{\gamma}_0) \otimes H^0(\gamma_1)^{(r)}) = 0$.
- (ii) $\text{Ext}_G^1(L(\widehat{\mu}_0) \otimes L(\mu_1)^{(r)}, I_{\widehat{\Gamma}_i}(\widehat{\gamma}_0) \otimes I_{\Gamma_i}(\gamma_1)^{(r)}) = 0$.
- (iii) $\text{Ext}_G^1(P_{\widehat{\Gamma}_i}(\widehat{\mu}_0) \otimes P_{\Gamma_i}(\mu_1)^{(r)}, L(\widehat{\gamma}_0) \otimes L(\gamma_1)^{(r)}) = 0$.

PROOF. We apply the LHS spectral sequence

$$(3.3.1) \quad E_2^{i,j} = \text{Ext}_{G/G_r}^i(V(\mu_1)^{(r)}, \text{Ext}_{G_r}^j(V(\widehat{\mu}_0), H^0(\widehat{\gamma}_0)) \otimes H^0(\gamma_1)^{(r)})$$

$$(3.3.2) \quad \Rightarrow \text{Ext}_G^{i+j}(V(\widehat{\mu}_0) \otimes V(\mu_1)^{(r)}, H^0(\widehat{\gamma}_0) \otimes H^0(\gamma_1)^{(r)}).$$

All weights involved are p^r restricted. Therefore,

$$\text{Hom}_{G_r}(V(\widehat{\mu}_0), H^0(\widehat{\gamma}_0)) \cong \text{Hom}_G(V(\widehat{\mu}_0), H^0(\widehat{\gamma}_0))$$

is either the trivial module or zero. It follows from [Jan1, II.4.13] that the $E_2^{1,0}$ -term vanishes.

For any composition factor $L(\lambda)$ of $V(\widehat{\mu}_0)^* \otimes H^0(\widehat{\gamma}_0)$, we have $\langle \lambda, \alpha_0^\vee \rangle \leq 2(h-1) - 1 + 2(h-1) - 1 = 4h-6$. Hence, by Lemma 3.2, $\text{Ext}_{G_r}^1(V(\widehat{\mu}_0), H^0(\widehat{\gamma}_0)) \cong \text{Ext}_G^1(V(\widehat{\mu}_0), H^0(\widehat{\gamma}_0))$ unless $p = 2$, Φ is of type C_n , and k is a composition factor of $V(\widehat{\mu}_0)^* \otimes H^0(\widehat{\gamma}_0)$. If we exclude this case, it follows from [Jan1, II.4.13] that the $E_2^{0,1}$ -term also vanishes.

If $p = 2$, Φ is of type C_n , and k is a composition factor of $V(\widehat{\mu}_0)^* \otimes H^0(\widehat{\gamma}_0)$ then $\mu_0 - w_0 \gamma_0$ is in the root lattice. Therefore, $\mu_1 - w_0 \gamma_1 = (w_0 \gamma_0 - w_0 \gamma_1) + (\mu_0 - w_0 \gamma_0) - (\mu_0 - \mu_1)$ is also in the root lattice. From Lemma 3.2, $\text{Ext}_{G_r}^1(V(\widehat{\mu}_0), H^0(\widehat{\gamma}_0)) \cong \text{Ext}_G^1(V(\widehat{\mu}_0), H^0(\widehat{\gamma}_0)) \oplus (\oplus_l L(\omega_1)^{(r)})$. By [Jan1, II.4.13], the first summand vanishes and so

$$E_2^{0,1} \cong \oplus_l \text{Hom}_{G/G_r}(V(\mu_1)^{(r)}, L(\omega_1)^{(r)} \otimes H^0(\gamma_1)^{(r)}).$$

Furthermore, this vanishes because

$$\begin{aligned} \text{Hom}_{G/G_r}(V(\mu_1)^{(r)}, L(\omega_1)^{(r)} \otimes H^0(\gamma_1)^{(r)}) \\ \cong \text{Hom}_G(L(-w_0 \omega_1), H^0(-w_0 \mu_1) \otimes H^0(\gamma_1)) \end{aligned}$$

and all weights of $H^0(-w_0 \mu_1)^{(r)} \otimes H^0(\gamma_1)$ are contained in the root lattice. The assertion (i) follows. Statements (ii) and (iii) follow along the same lines. \square

REMARK . We assume that the conditions of Proposition 3.3 are satisfied and have a closer look at (3.3.1). It follows from [Jan1, II.4.13] and Lemma 3.2 that $E_2^{i,0} = 0$ for $i > 0$ and $E_2^{i,1} = 0$ for $i \geq 0$. Hence $E^2 \cong E_2^{0,2}$. If $\mu_0 = \gamma_0 = \gamma_1 = 0$, then $E_2^{0,2} = \text{Hom}_{G/G_r}(V(\mu_1)^r, \text{Ext}_{G_r}^2(k, k))$. The group $\text{Ext}_{G_r}^2(k, k)$ does not vanish [BNP6] and for appropriate choices of μ_1 neither does the E^2 -term. We conclude that neither

$$\text{Ext}_G^2(V(\widehat{\mu}_0) \otimes V(\mu_1)^{(r)}, H^0(\widehat{\gamma}_0) \otimes H^0(\gamma_1)^{(r)})$$

nor

$$\text{Ext}_G^2(L(\widehat{\mu}_0) \otimes L(\mu_1)^{(r)}, I_{\widehat{\Gamma}_i}(\widehat{\gamma}_0) \otimes I_{\Gamma_i}(\gamma_1)^{(r)})$$

vanish in general.

3.4. Important submodules for injectives. It follows from the Remark in Section 3.3 that $I_{\widehat{\Gamma}_i}(\widehat{\gamma}_0) \otimes I_{\Gamma_i}(\gamma_1)^{(r)}$ is not injective for the full subcategory of modules whose composition factors are of the form $L(\widehat{\nu}_0 + p^r \nu_1)$ with $\nu_0, \nu_1 \in \Gamma_i$. Perhaps this is not surprising since the set of weights $\{\widehat{\nu}_0 + p^r \nu_1 \mid \nu_0, \nu_1 \in \Gamma_i\}$ together with its W -conjugates is in general not saturated. However, under the conditions of Proposition 3.3, the module $I_{\widehat{\Gamma}_i}(\widehat{\gamma}_0) \otimes I_{\Gamma_i}(\gamma_1)^{(r)}$ can be characterized as the maximal submodule of $I(\widehat{\nu}_0 + p^r \nu_1)$ whose composition factors are of the form $L(\widehat{\nu}_0 + p^r \nu_1)$ with $\nu_0, \nu_1 \in \Gamma_i$. This will be shown in the following lemma.

Assume that $p^r > 2(h-1)$ and $\gamma_0, \gamma_1 \in \Gamma$. Then all composition factors of $I_\Gamma(\widehat{\gamma}_0)$ have p^r -restricted highest weight. Therefore, $\text{soc}_{G_r}(I_\Gamma(\widehat{\gamma}_0)) \cong L(\widehat{\gamma}_0)$ and $\text{soc}_{G_r}(I_\Gamma(\widehat{\gamma}_0) \otimes I_\Gamma(\gamma_1)^{(r)}) \cong L(\widehat{\gamma}_0) \otimes I_\Gamma(\gamma_1)^{(r)}$. It follows from [Jan1, II.3.16(2)] that

$$\text{soc}_G(I_\Gamma(\widehat{\gamma}_0) \otimes I_\Gamma(\gamma_1)^{(r)}) \cong L(\widehat{\gamma}_0) \otimes \text{soc}_G(I_\Gamma(\gamma_1)^{(r)}) \cong L(\widehat{\gamma}_0 + p^r \gamma_1).$$

Therefore,

$$(3.4.1) \quad I_\Gamma(\widehat{\gamma}_0) \otimes I_\Gamma(\gamma_1)^{(r)} \hookrightarrow I(\widehat{\gamma}_0 + p^r \gamma_1), \text{ and } P_\Omega(\widehat{\gamma}_0 + p^r \gamma_1) \twoheadrightarrow P_\Gamma(\widehat{\gamma}_0) \otimes P_\Gamma(\gamma_1)^{(r)}.$$

LEMMA . Assume that the root system Φ of G is not of type A_1 . Let $p^{r-1}(p-1) > 4h-6$ and let M be a finite dimensional rational G -module whose composition factors are of the form $L(\widehat{\gamma}_0 + p^r \gamma_1)$ with $\gamma_0, \gamma_1 \in \Gamma_i$. If $p = 2$ and Φ is of type C_n , we assume in addition that $\gamma_0 - \gamma_1$ is contained in the root lattice. Then

$$\begin{aligned} [M : L(\widehat{\gamma}_0 + p^r \gamma_1)]_G &= \dim \text{Hom}_G(M, I_{\widehat{\Gamma}_i}(\widehat{\gamma}_0) \otimes I_{\Gamma_i}(\gamma_1)^{(r)}) \\ &= \dim \text{Hom}_G(P_{\widehat{\Gamma}_i}(\widehat{\gamma}_0) \otimes P_{\Gamma_i}(\gamma_1)^{(r)}, M). \end{aligned}$$

PROOF. The second equality follows easily by using duality. We will proceed to prove the first equality. Clearly, $[M : L(\widehat{\gamma}_0 + p^r \gamma_1)]_G = \dim \text{Hom}_G(M, I(\widehat{\gamma}_0 + p^r \gamma_1))$. It suffices to prove that

$$\dim \text{Hom}_G(M, I(\widehat{\gamma}_0 + p^r \gamma_1)) = \dim \text{Hom}_G(M, I_{\widehat{\Gamma}_i}(\widehat{\gamma}_0) \otimes I_{\Gamma_i}(\gamma_1)^{(r)}).$$

Clearly, for $p = 2$ and Φ of type C_n both Hom-groups vanish unless $\gamma_0 - \gamma_1$ is contained in the root lattice. We use induction on the number of composition factors of M . The module $I_{\widehat{\Gamma}_i}(\widehat{\gamma}_0) \otimes I_{\Gamma_i}(\gamma_1)^{(r)}$ is a submodule of $I(\widehat{\gamma}_0 + p^r \gamma_1)$ (by (3.4.1)) and their socles are simple. The assertion holds therefore for simple modules. Next assume that $L(\widehat{\nu}_0 + p^r \nu_1)$ is a simple quotient of M . This implies for the case $p = 2$ and Φ of type C_n that $\nu_0 - \nu_1$ is in the root lattice. Define S via

the exact sequence $0 \rightarrow S \rightarrow M \rightarrow L(\widehat{\nu}_0 + p^r \nu_1) \rightarrow 0$. One obtains the long exact sequences

$$\begin{aligned} 0 &\rightarrow \operatorname{Hom}_G(L(\widehat{\nu}_0 + p^r \nu_1), I(\widehat{\gamma}_0 + p^r \gamma_1)) \rightarrow \operatorname{Hom}_G(M, I(\widehat{\gamma}_0 + p^r \gamma_1)) \\ &\rightarrow \operatorname{Hom}_G(S, I(\widehat{\gamma}_0 + p^r \gamma_1)) \rightarrow \operatorname{Ext}_G^1(L(\widehat{\nu}_0 + p^r \nu_1), I(\widehat{\gamma}_0 + p^r \gamma_1)) \end{aligned}$$

and

$$\begin{aligned} 0 &\rightarrow \operatorname{Hom}_G(L(\widehat{\nu}_0 + p^r \nu_1), I_{\widehat{\Gamma}_i}(\widehat{\gamma}_0) \otimes I_{\Gamma_i}(\gamma_1)^{(r)}) \rightarrow \operatorname{Hom}_G(M, I_{\widehat{\Gamma}_i}(\widehat{\gamma}_0) \otimes I_{\Gamma_i}(\gamma_1)^{(r)}) \\ &\rightarrow \operatorname{Hom}_G(S, I_{\widehat{\Gamma}_i}(\widehat{\gamma}_0) \otimes I_{\Gamma_i}(\gamma_1)^{(r)}) \rightarrow \operatorname{Ext}_G^1(L(\widehat{\nu}_0 + p^r \nu_1), I_{\widehat{\Gamma}_i}(\widehat{\gamma}_0) \otimes I_{\Gamma_i}(\gamma_1)^{(r)}). \end{aligned}$$

Obviously, $\operatorname{Ext}_G^1(L(\widehat{\nu}_0 + p^r \nu), I(\widehat{\gamma}_0 + p^r \gamma_1)) = 0$. Moreover, Proposition 3.3 shows that $\operatorname{Ext}_G^1(L(\widehat{\nu}_0 + p^r \nu), I_{\widehat{\Gamma}_i}(\widehat{\gamma}_0) \otimes I_{\Gamma_i}(\gamma_1)^{(r)}) = 0$. From the induction hypotheses one concludes that

$$\dim \operatorname{Hom}_G(L(\widehat{\nu}_0 + p^r \nu), I(\widehat{\gamma}_0 + p^r \gamma_1)) = \dim \operatorname{Hom}_G(L(\widehat{\nu}_0 + p^r \nu), I_{\widehat{\Gamma}_i}(\widehat{\gamma}_0) \otimes I_{\Gamma_i}(\gamma_1)^{(r)})$$

and

$$\dim \operatorname{Hom}_G(S, I(\widehat{\gamma}_0 + p^r \gamma_1)) = \dim \operatorname{Hom}_G(S, I_{\widehat{\Gamma}_i}(\widehat{\gamma}_0) \otimes I_{\Gamma_i}(\gamma_1)^{(r)}).$$

Hence, $\dim \operatorname{Hom}_G(M, I(\widehat{\gamma}_0 + p^r \gamma_1)) = \dim \operatorname{Hom}_G(M, I_{\widehat{\Gamma}_i}(\widehat{\gamma}_0) \otimes I_{\Gamma_i}(\gamma_1)^{(r)})$ as claimed. \square

4. A Filtration of $\mathcal{G}_\Omega(k)$

The tensor identity says that $\mathcal{G}(\operatorname{St}_r) \cong \operatorname{St}_r \otimes \mathcal{G}(k)$. Once we pass to truncated categories, such an identity no longer holds in general. However, for $p^r \geq 4(h-1)$ we will show that $\mathcal{G}_\Lambda(\operatorname{St}_r) \cong \operatorname{St}_r \otimes \mathcal{G}_\Omega(k)$ (Proposition 4.4). Furthermore, a complete description of the module $\mathcal{G}_\Lambda(\operatorname{St}_r)$ will be given (Theorem 4.3). This will allow us to determine the character of $\mathcal{G}_\Omega(k)$. The ultimate goal of this section is to study the filtration of $\mathcal{G}_{\Omega_n}(k)$ given in (2.2.1) in order to identify the factors and show that $\mathcal{G}_{\Omega_n}(k) = \mathcal{G}_\Omega(k)$ for sufficiently large p^r (Theorem 4.7).

4.1. Composition factors of $\mathcal{G}_\Omega(k)$. Since $-w_0$ and σ permute the fundamental weights, any weight $\gamma \in X(T)_+$ can be expressed uniquely in the form $\gamma = -w_0 \sigma \gamma_0 + p^r \gamma_1 = \widehat{\gamma}_0 + p^r \gamma_1$ with $\gamma_0 \in X_r(T)$ and $\gamma_1 \in X(T)_+$. One can now use the methods in [BNP5, Prop. 2.5] to prove the following result.

PROPOSITION . *If $L(\widehat{\gamma}_0 + p^r \gamma_1)$ is a composition factor of $\mathcal{G}_\Omega(k)$ then $\gamma_0, \gamma_1 \in \Gamma$.*

4.2. The following results will help us relate $\mathcal{G}_\Omega(k)$ to $\mathcal{G}_\Lambda(\operatorname{St}_r)$ and understand $\mathcal{G}_\Lambda(\operatorname{St}_r)$.

PROPOSITION (A). *Let $p^r \geq 4(h-1)$ and $\nu_0, \nu_1 \in \Gamma$.*

- (a) $\operatorname{St}_r \otimes L(\nu_0) \otimes L(\nu_1)^{(r)} \in \operatorname{Mod}(\Lambda)$;
- (b) $\operatorname{St}_r \otimes L(\nu_0) \otimes I_\Gamma(\nu_1)^{(r)} \in \operatorname{Mod}(\Lambda)$;
- (c) $\operatorname{St}_r \otimes \mathcal{G}_\Omega(k)$ is a submodule of $\mathcal{G}_\Lambda(\operatorname{St}_r)$.

PROOF. (a) From $\nu_1 \in \Gamma$ one concludes that $\langle \nu_1, \alpha_0^\vee \rangle \leq 2(h-1) - 1$. Therefore,

$$\begin{aligned} \langle (p^r - 1)\rho + \nu_0 + p^r \nu_1, \alpha_0^\vee \rangle &< (p^r - 1)(h - 1) + 2(h - 1) + p^r 2(h - 1) - p^r \\ &= 3(p^r - 1)(h - 1) + 4(h - 1) - p^r \\ &\leq 3(p^r - 1)(h - 1). \end{aligned}$$

Part (b) follows immediately from above. Proposition 4.1 and part (a) imply that $\text{St}_r \otimes \mathcal{G}_\Omega(k) \in \text{Mod}(\Lambda)$. Part (c) now follows from $\text{St}_r \otimes \mathcal{G}_\Omega(k) \subset (\text{St}_r \otimes \mathcal{G}(k))_\Lambda \cong \mathcal{G}_\Lambda(\text{St}_r)$. \square

PROPOSITION (B). *Let $p^r \geq 4(h-1)$ and $\nu_0, \nu_1 \in \Gamma$. Then*

- (a) $\text{St}_r \otimes L(\nu_0) \otimes I_\Gamma(\nu_1)^{(r)}$ is an injective module in $\text{Mod}(\Lambda)$.
- (b) $\text{soc}_G(\text{St}_r \otimes L(\nu_0) \otimes I_\Gamma(\nu_1)^{(r)}) \cong \text{soc}_G(\text{St}_r \otimes L(\nu_0)) \otimes L(\nu_1)^{(r)}$

PROOF. (a) Let $\gamma = \gamma_0 + p^r \gamma_1$ be a weight in Λ . It suffices to show that $\text{Ext}_G^1(L(\gamma), \text{St}_r \otimes L(\nu_0) \otimes I_\Gamma(\nu_1)^{(r)}) = 0$ for all such γ . Consider the LHS spectral sequence

$$\begin{aligned} E_2^{i,j} &= \text{Ext}_{G/G_r}^i(L(\gamma_1)^{(r)}, \text{Ext}_{G_r}^j(L(\gamma_0), \text{St}_r \otimes L(\nu_0)) \otimes I_\Gamma(\nu_1)^{(r)}) \\ &\Rightarrow \text{Ext}_G^{i+j}(L(\gamma_0) \otimes L(\gamma_1)^{(r)}, \text{St}_r \otimes L(\nu_0) \otimes I_\Gamma(\nu_1)^{(r)}). \end{aligned}$$

All $E^{i,j}$ -terms with $j > 0$ vanish because St_r is injective as a G_r -module. Thus

$$\begin{aligned} E^1 \cong E_2^{1,0} &\cong \text{Ext}_{G/G_r}^1(L(\gamma_1)^{(r)}, \text{Hom}_{G_r}(L(\gamma_0), \text{St}_r \otimes L(\nu_0)) \otimes I_\Gamma(\nu_1)^{(r)}) \\ &\cong \text{Ext}_{G/G_r}^1(L(\gamma_1)^{(r)}, I_\Gamma(\nu_1)^{(r)}) \otimes \text{Hom}_G(L(\gamma_0), \text{St}_r \otimes L(\nu_0)). \end{aligned}$$

The last isomorphism is a consequence of Proposition 3.1. For $\text{Hom}_G(L(\gamma_0), \text{St}_r \otimes L(\nu_0))$ to be non-zero, it is necessary that $\langle \gamma_0 - w_0 \nu_0, \alpha_0^\vee \rangle \geq (p^r - 1)(h - 1)$. This forces $\langle \gamma_0, \alpha_0^\vee \rangle > (p^r - 1)(h - 1) - 2(h - 1)$ and

$$(p^r - 1)(h - 1) - 2(h - 1) + p^r \langle \gamma_1, \alpha_0^\vee \rangle \leq \langle \gamma, \alpha_0^\vee \rangle < 3(p^r - 1)(h - 1).$$

One concludes that $p^r \langle \gamma_1, \alpha_0^\vee \rangle < 2p^r(h - 1)$ and $\gamma_1 \in \Gamma$. Hence, by the injectivity of $I_\Gamma(\nu_1)$, $\text{Ext}_{G/G_r}^1(L(\gamma_1)^{(r)}, I_\Gamma(\nu_1)^{(r)})$ vanishes.

(b) Let $\gamma = \gamma_0 + p^r \gamma_1$ be a weight in Λ . From Proposition 3.1, one concludes that

$$\begin{aligned} &\text{Hom}_G(L(\gamma), \text{St}_r \otimes L(\nu_0) \otimes I_\Gamma(\nu_1)^{(r)}) \\ &\cong \text{Hom}_{G/G_r}(L(\gamma_1)^{(r)}, \text{Hom}_{G_r}(L(\gamma_0), \text{St}_r \otimes L(\nu_0)) \otimes I_\Gamma(\nu_1)^{(r)}) \\ &\cong \text{Hom}_{G/G_r}(L(\gamma_1)^{(r)}, I_\Gamma(\nu_1)^{(r)}) \otimes \text{Hom}_G(L(\gamma_0), \text{St}_r \otimes L(\nu_0)). \end{aligned}$$

The assertion follows. \square

4.3. We can now provide a description of $\mathcal{G}_\Lambda(\text{St}_r)$ for p^r sufficiently large.

THEOREM . *Let $p^r \geq 4(h-1)$. Then*

$$\mathcal{G}_\Lambda(\text{St}_r) \cong \bigoplus_{\nu \in \Gamma} \text{St}_r \otimes L(\hat{\nu}) \otimes I_\Gamma(\nu)^{(r)}.$$

PROOF. Both modules are injective in $\text{Mod}(\Lambda)$. (The left-side since St_r is injective over $G(\mathbb{F}_q)$ and the right-side by Proposition 4.2(B) part (a).) It suffices therefore to show that both modules have the same G -socle. Let $\gamma = \gamma_0 + p^r \gamma_1$ be a weight in Λ .

$$\begin{aligned} \dim \text{Hom}_G(L(\gamma), \mathcal{G}_\Lambda(\text{St}_r)) &= \dim \text{Hom}_G(L(\gamma_0) \otimes L(\gamma_1)^{(r)}, \text{St}_r \otimes \mathcal{G}(k)) \\ &= \dim \text{Hom}_{G(\mathbb{F}_q)}(L(\gamma_0) \otimes L(\sigma \gamma_1), \text{St}_r) \text{ (by adjointness)} \\ &= [L(\gamma_0) \otimes L(\sigma \gamma_1) : \text{St}_r]_{G(\mathbb{F}_q)} \\ &= \sum_{\mu \in X(T)^+} [L(\gamma_0) \otimes L(\sigma \gamma_1) \otimes L(\sigma \mu) : \text{St}_r \otimes L(\mu)^{(r)}]_G, \end{aligned}$$

where the last equality follows from [Jan2, Satz 1.5]. The above expression is zero unless

$$(4.3.1) \quad \langle \gamma_0 + \gamma_1, \alpha_0^\vee \rangle \geq (p^r - 1)(h - 1) + (p^r - 1)\langle \mu, \alpha_0^\vee \rangle.$$

Moreover,

$$3(p^r - 1)(h - 1) > \langle \gamma_0 + p^r \gamma_1, \alpha_0^\vee \rangle \geq (p^r - 1)(h - 1) + (p^r - 1)\langle \mu, \alpha_0^\vee \rangle + (p^r - 1)\langle \gamma_1, \alpha_0^\vee \rangle.$$

The last inequality implies that $\langle \gamma_1, \alpha_0^\vee \rangle < 2(h - 1)$. Hence, any weight γ that appears in the socle of $\mathcal{G}_\Lambda(\text{St}_r)$ has γ_1 contained in Γ .

It follows now from (4.3.1) that $2(h - 1) > (p^r - 1)\langle \mu, \alpha_0^\vee \rangle$. Together with $p^r \geq 4(h - 1)$ this forces $\mu = 0$. One concludes that, if $L(\gamma)$ is a composition factor of $\mathcal{G}_\Lambda(\text{St}_r)$, then $\dim \text{Hom}_G(L(\gamma), \mathcal{G}_\Lambda(\text{St}_r)) = [L(\gamma_0) \otimes L(\sigma\gamma_1) : \text{St}_r]_G$.

Using Proposition 3.1 and the fact that St_r is injective as a G_r -module, one observes that

$$\begin{aligned} \dim \text{Hom}_G(L(\gamma_0) \otimes L(\sigma\gamma_1) : \text{St}_r) &\leq [L(\gamma_0) \otimes L(\sigma\gamma_1) : \text{St}_r]_G \\ &\leq [L(\gamma_0) \otimes L(\sigma\gamma_1) : \text{St}_r]_{G_r} \\ &= \dim \text{Hom}_{G_r}(L(\gamma_0) \otimes L(\sigma\gamma_1) : \text{St}_r) \\ &= \dim \text{Hom}_G(L(\gamma_0) \otimes L(\sigma\gamma_1) : \text{St}_r). \end{aligned}$$

Therefore,

$$\begin{aligned} \dim \text{Hom}_G(L(\gamma), \mathcal{G}_\Lambda(\text{St}_r)) &= \dim \text{Hom}_G(L(\gamma_0), \text{St}_r \otimes L(-w_0\sigma\gamma_1)) \\ &= \dim \text{Hom}_G(L(\gamma_0), \text{St}_r \otimes L(\hat{\gamma}_1)) \end{aligned}$$

and

$$\begin{aligned} \text{soc}_G(\mathcal{G}_\Lambda(\text{St}_r)) &\cong \bigoplus_{\gamma_0 \in X_r(T), \gamma_1 \in \Gamma} \text{Hom}_G(L(\gamma_0), \text{St}_r \otimes L(\hat{\gamma}_1)) \otimes L(\gamma_0) \otimes L(\gamma_1)^{(r)} \\ &\cong \bigoplus_{\gamma_1 \in \Gamma} \text{soc}_G(\text{St}_r \otimes L(\hat{\gamma}_1)) \otimes L(\gamma_1)^{(r)} \\ &\cong \bigoplus_{\nu \in \Gamma} \text{soc}_G(\text{St}_r \otimes L(\hat{\nu})) \otimes I_\Gamma(\nu)^{(r)} \end{aligned}$$

by Proposition 4.2(B) part (b). □

4.4. A “tensor identity” and the character of $\mathcal{G}_\Omega(k)$.

PROPOSITION . *Let $p^r \geq 4(h - 1)$. Then*

$$\mathcal{G}_\Lambda(\text{St}_r) \cong \text{St}_r \otimes \mathcal{G}_\Omega(k) \text{ and } \text{ch } \mathcal{G}_\Omega(k) = \sum_{\lambda \in \Gamma} \text{ch}(H^0(\hat{\lambda}) \otimes H^0(\lambda)^{(r)}).$$

PROOF. From Proposition 4.2(A), we know that $\text{St}_r \otimes \mathcal{G}_\Omega(k) \subset \mathcal{G}_\Lambda(\text{St}_r)$. We will show here that equality holds. From Theorem 4.3, we know that the formal character of $\mathcal{G}_\Omega(k)$ is a summand of $\text{ch}(\bigoplus_{\nu \in \Gamma} L(\hat{\nu}) \otimes I_\Gamma(\nu)^{(r)})$ and, by Proposition 4.1, all composition factors of $\mathcal{G}_\Omega(k)$ are of the form $L(\hat{\gamma}_0) \otimes L(\gamma_1)^{(r)}$ with $\gamma_0, \gamma_1 \in \Gamma$.

Note that since $\sigma(\alpha_0) = \alpha_0$ and $-w_0(\alpha_0) = \alpha_0$, $\sigma(\Gamma) = \Gamma = -w_0(\Gamma)$. We have

$$\begin{aligned}
[\mathcal{G}_\Omega(k) : L(\widehat{\gamma}_0 + p^r \gamma_1)]_G &= \dim \operatorname{Hom}_G(P_\Omega(\widehat{\gamma}_0 + p^r \gamma_1), \mathcal{G}_\Omega(k)) \\
&\geq \dim \operatorname{Hom}_G(P_\Gamma(\widehat{\gamma}_0) \otimes P_\Gamma(\gamma_1)^{(r)}, \mathcal{G}_\Omega(k)) \quad (\text{by 3.4.1}) \\
&= \dim \operatorname{Hom}_{G_\sigma(\mathbb{F}_q)}(P_\Gamma(\widehat{\gamma}_0) \otimes P_\Gamma(\gamma_1)^{(r)}, k) \quad (\text{by adjointness}) \\
&= \dim \operatorname{Hom}_{G_\sigma(\mathbb{F}_q)}(P_\Gamma(\widehat{\gamma}_0) \otimes P_\Gamma(\sigma\gamma_1), k) \quad (\text{by 2.3.4}) \\
&= \dim \operatorname{Hom}_G(P_\Gamma(\widehat{\gamma}_0) \otimes P_\Gamma(\sigma\gamma_1), k) \quad (\text{by [CPSvdK, Thm. 7.4]}) \\
&= \dim \operatorname{Hom}_G(P_\Gamma(\sigma\gamma_1), I_\Gamma(\sigma\gamma_0)) \quad (\text{by duality}) \\
&= \dim \operatorname{Hom}_G(P_\Gamma(\gamma_1), I_\Gamma(\gamma_0)) \quad (\text{by 2.3.3}) \\
&= [I_\Gamma(\gamma_0) : L(\gamma_1)]_G \\
&= [(L(\widehat{\gamma}_0) \otimes I_\Gamma(\gamma_0)^{(r)}) : L(\widehat{\gamma}_0) \otimes L(\gamma_1)^{(r)}]_G \\
&= [(\bigoplus_{\nu \in \Gamma} L(\widehat{\nu}) \otimes I_\Gamma(\nu)^{(r)}) : L(\widehat{\gamma}_0 + p^r \gamma_1)]_G.
\end{aligned}$$

It follows that $\mathcal{G}_\Lambda(\operatorname{St}_r) \cong \operatorname{St}_r \otimes \mathcal{G}_\Omega(k)$. Finally,

$$\begin{aligned}
\operatorname{ch}(\bigoplus_{\nu \in \Gamma} L(\widehat{\nu}) \otimes I_\Gamma(\nu)^{(r)}) &= \sum_{\nu \in \Gamma} \operatorname{ch}(L(\widehat{\nu}) \otimes I_\Gamma(\nu)^{(r)}) \\
&= \sum_{\nu \in \Gamma} \sum_{\lambda \in \Gamma} [I_\Gamma(\nu) : H^0(\lambda)] \operatorname{ch}(L(\widehat{\nu}) \otimes H^0(\lambda)^{(r)}) \\
&= \sum_{\lambda \in \Gamma} \sum_{\nu \in \Gamma} [H^0(\widehat{\lambda}) : L(\widehat{\nu})] \operatorname{ch}(L(\widehat{\nu}) \otimes H^0(\lambda)^{(r)}) \\
&= \sum_{\lambda \in \Gamma} \operatorname{ch}(H^0(\widehat{\lambda}) \otimes H^0(\lambda)^{(r)}).
\end{aligned}$$

□

We immediately get the following where \uparrow is the ordering on $X(T)$ as given in [Jan1, II.6.4].

COROLLARY . *Let $p^r \geq 4(h-1)$. If $L(\widehat{\gamma}_0 + p^r \gamma_1)$ is a composition factor of $\mathcal{G}_\Omega(k)$, then there exists $\lambda \in \Gamma$ such that $\gamma_0 \uparrow \lambda$ and $\gamma_1 \uparrow \lambda$.*

REMARK . An easy computation shows for $p^r > 4(h-1)$ that $\mathcal{H}_\Omega(k) \cong \bigoplus_{\nu \in X(T)_+} (I_\Gamma(\nu)^{(r)})^{\dim L(\nu)}$ and $\mathcal{H}_\Lambda(\operatorname{St}_r) \cong \operatorname{St}_r \otimes \mathcal{H}_\Omega(k)$. One concludes from Theorem 4.3 and Proposition 4.4 that

$$\dim \mathcal{H}_\Lambda(\operatorname{St}_r) = \dim \mathcal{G}_\Lambda(\operatorname{St}_r) \text{ and } \dim \mathcal{H}_\Omega(k) = \dim \mathcal{G}_\Omega(k).$$

4.5. Good p^r -filtrations and Donkin's conjecture. Let M be a finite dimensional G -module. We say that M has a good p^r -filtration if and only if there exists a sequence of submodules $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_s = M$ such that $M_i/M_{i+1} \cong L(\mu_i^0) \otimes H^0(\mu_i^1)^{(r)}$ where $\mu_i = \mu_i^0 + p^r \mu_i^1 \in X(T)_+$ with $\mu_i^0 \in X_r(T)$ for all i .

The definition for the case when $r = 1$ was first introduced by Donkin in 1990 (i.e. the notion of a good p -filtration). Donkin conjectured that if M is a finite dimensional rational G -module, then M has a good p -filtration if and only if $M \otimes \operatorname{St}_1$ has a good filtration. Andersen [And2, Cor. 2.8] proved one direction of

a generalization of this conjecture for large primes ($p \geq 2(h-1)$), namely, given a finite-dimensional rational G -module M with a good p^r -filtration then $M \otimes \text{St}_r$ has a good filtration. In the same paper, it is also shown that the other direction holds for $G = SL_2(k)$.

Since St_r is injective over $G_\sigma(\mathbb{F}_q)$ and Λ is saturated, $\mathcal{G}_\Lambda(\text{St}_r)$ has a good filtration. For $p^r \geq 4(h-1)$, from Proposition 4.4, it follows that $\text{St}_r \otimes \mathcal{G}_\Omega(k)$ has a good filtration. The validity of the generalized version of Donkin's conjecture would imply that $\mathcal{G}_\Omega(k)$ has a good p^r -filtration. One can see some indication of this in Proposition 4.4. Indeed, this provides the motivation for formulating and proving Theorem 4.7 which demonstrates in the case when $M = \mathcal{G}_\Omega(k)$ and $p^{r-1}(p-1) > 4h-6$ that the generalized version of Donkin's conjecture holds.

4.6. The G -socle of $\mathcal{G}_{\Omega_i}(k)$. We now determine the socle of each $\mathcal{G}_{\Omega_i}(k)$ for p^r sufficiently large.

PROPOSITION . *Let $p^r > 2(h-1)$ and $\nu_0, \nu_1 \in X_r(T)$. Then*

$$\text{Hom}_G(V(\widehat{\nu}_0) \otimes V(\nu_1)^{(r)}, \mathcal{G}_{\Omega_i}(k)) \cong \begin{cases} k & \text{if } \nu_0 = \nu_1 \text{ and } \nu_1 \in \Gamma_i, \\ 0 & \text{else.} \end{cases}$$

PROOF. Set $N = V(\widehat{\nu}_0) \otimes V(\nu_1)^{(r)}$. Then

$$\begin{aligned} \text{Hom}_G(N, \mathcal{G}_{\Omega_i}(k)) &\subseteq \text{Hom}_G(N, \text{ind}_{G_\sigma(\mathbb{F}_q)}^G(k)) \\ &= \text{Hom}_{G_\sigma(\mathbb{F}_q)}(N, k) \text{ (by adjointness)} \\ &= \text{Hom}_{G_\sigma(\mathbb{F}_q)}(V(\widehat{\nu}_0) \otimes V(\sigma\nu_1), k) \\ &= \text{Hom}_{G_\sigma(\mathbb{F}_q)}(V(\sigma\nu_1), H^0(\sigma\nu_0)) \\ &= \text{Hom}_G(V(\sigma\nu_1), H^0(\sigma\nu_0)) \text{ (since } p^r > 2(h-1)) \\ &= \begin{cases} k & \text{if } \nu_0 = \nu_1 \\ 0 & \text{else.} \end{cases} \end{aligned}$$

If $\nu_0 = \nu_1$, then a non-trivial homomorphism will exist precisely if $(p^r - w_0\sigma)\nu_1 = \widehat{\nu}_0 + p^r\nu_1 \in \Omega_i$. That occurs if and only if $\nu_0 = \nu_1 \in \Gamma_i$. \square

COROLLARY . *Let $p^r > 2(h-1)$. Then*

$$\text{soc}_G \mathcal{G}_{\Omega_i}(k) = \bigoplus_{\nu \in \Gamma_i} L(\widehat{\nu} + p^r\nu).$$

PROOF. Any simple G -module $L(\widehat{\nu} + p^r\nu)$ in the socle of $\mathcal{G}_{\Omega_i}(k)$ gives rise to a non-trivial homomorphism from $V(\widehat{\mu}) \otimes V(\nu)^{(r)}$ to $\mathcal{G}_{\Omega_i}(k)$. The result now follows from the proposition. \square

REMARK . From Proposition 4.4, Corollary 4.6 (for $p^r \geq 4$) and our earlier results in [BNP2] (for $p = 3$) we conclude the following:

$$(4.6.1) \quad \text{If } G \text{ is of type } A_1 \text{ and } p^r > 2, \text{ then } \mathcal{G}_\Omega(k) \cong k \oplus L(p^r + 1).$$

4.7. Filtrations of $\mathcal{G}_{\Omega_i}(k)$ and $\mathcal{G}_{\Omega}(k)$. In this section, we show that $\mathcal{G}_{\Omega}(k)$ admits a natural filtration.

THEOREM . *Let $p^{r-1}(p-1) > 4h-6$. Then $\mathcal{G}_{\Omega}(k)$ has a filtration*

$$0 = \mathcal{G}_{\Omega_0}(k) \subseteq \mathcal{G}_{\Omega_1}(k) \subseteq \mathcal{G}_{\Omega_2}(k) \subseteq \cdots \subseteq \mathcal{G}_{\Omega_n}(k) = \mathcal{G}_{\Omega}(k)$$

with factors $\mathcal{G}_{\Omega_j}(k)/\mathcal{G}_{\Omega_{j-1}}(k) \cong H^0(\widehat{\lambda}_j) \otimes H^0(\lambda_j)^{(r)}$, each $\lambda_j \in \Gamma$ appearing exactly once.

PROOF. The submodules $\mathcal{G}_{\Omega_i}(k)$ were defined in (2.2.1). Using induction on i , we will show that $\mathcal{G}_{\Omega_i}(k)/\mathcal{G}_{\Omega_{i-1}}(k) \cong H^0(\widehat{\lambda}_i) \otimes H^0(\lambda_i)^{(r)}$. For $i=1$ one has $\Omega_1 = \Gamma_1 = \{0\}$ and $\mathcal{G}_{\Omega_1}(k) \cong k \cong H^0(0) \otimes H^0(0)^{(r)}$. Assume $i > 1$. By Proposition 4.1 and Corollary 4.4, all composition factors of $\mathcal{G}_{\Omega_i}(k) \subset \mathcal{G}_{\Omega}(k)$ are of the form $L(\widehat{\nu}_0 + p^r \nu_1)$ with $\nu_0, \nu_1 \in \Gamma$ and $\nu_0 - \nu_1$ in the root lattice. We first show inductively on i that in fact $\nu_0, \nu_1 \in \Gamma_i$. Note that this clearly holds for $i=0$.

The Theorem holds for type A_1 by (4.6.1). For the remainder of the proof we may therefore assume that G is not of type A_1 . We apply Proposition 3.3 to conclude that $\text{Ext}_G^1(V(\widehat{\nu}_0) \otimes V(\nu_1)^{(r)}, H^0(\widehat{\lambda}_j) \otimes H^0(\lambda_j)^{(r)}) = 0$ for $\nu_0, \nu_1, \lambda_j \in \Gamma$. The induction hypothesis implies that $\text{Ext}_G^1(V(\widehat{\nu}_0) \otimes V(\nu_1)^{(r)}, \mathcal{G}_{\Omega_{i-1}}(k)) = 0$. Therefore the short exact sequence

$$0 \rightarrow \mathcal{G}_{\Omega_{i-1}}(k) \rightarrow \mathcal{G}_{\Omega_i}(k) \rightarrow \mathcal{G}_{\Omega_i}(k)/\mathcal{G}_{\Omega_{i-1}}(k) \rightarrow 0$$

gives rise to the exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}_G(V(\widehat{\nu}_0) \otimes V(\nu_1)^{(r)}, \mathcal{G}_{\Omega_{i-1}}(k)) &\rightarrow \text{Hom}_G(V(\widehat{\nu}_0) \otimes V(\nu_1)^{(r)}, \mathcal{G}_{\Omega_i}(k)) \\ &\rightarrow \text{Hom}_G(V(\widehat{\nu}_0) \otimes V(\nu_1)^{(r)}, \mathcal{G}_{\Omega_i}(k)/\mathcal{G}_{\Omega_{i-1}}(k)) \rightarrow 0. \end{aligned}$$

One concludes from Proposition 4.6 that

$$\text{Hom}_G(V(\widehat{\nu}_0) \otimes V(\nu_1)^{(r)}, \mathcal{G}_{\Omega_i}(k)/\mathcal{G}_{\Omega_{i-1}}(k)) \cong \begin{cases} k & \text{if } \nu_0 = \nu_1 = \lambda_i, \\ 0 & \text{else.} \end{cases}$$

The module $V(\widehat{\lambda}_i) \otimes V(\lambda_i)^{(r)}$ has simple G -head $L(\widehat{\lambda}_i + p^r \lambda_i)$ and one concludes that the G -socle of $\mathcal{G}_{\Omega_i}(k)/\mathcal{G}_{\Omega_{i-1}}(k)$ is isomorphic to $L(\widehat{\lambda}_i + p^r \lambda_i)$. Next we embed $\mathcal{G}_{\Omega_i}(k)/\mathcal{G}_{\Omega_{i-1}}(k)$ in the injective hull $I(\widehat{\lambda}_i + p^r \lambda_i)$. The module $I(\widehat{\lambda}_i + p^r \lambda_i)$ has a good filtration with factors $H^0(\gamma)$ with $(p^r - w_0\sigma)\lambda_i \uparrow \gamma$. Here \uparrow is the ordering on $X(T)$ as given in [Jan1, II.6.4]. Clearly the only such γ that is contained in Ω_i is $(p^r - w_0\sigma)\lambda_i$. One obtains an embedding $\mathcal{G}_{\Omega_i}(k)/\mathcal{G}_{\Omega_{i-1}}(k) \hookrightarrow H^0((p^r - w_0\sigma)\lambda_i)$. Therefore all weights γ in $\mathcal{G}_{\Omega_i}(k)/\mathcal{G}_{\Omega_{i-1}}(k)$ satisfy $\gamma \uparrow (p^r - w_0\sigma)\lambda_i$.

From above, the multiplicity $[\mathcal{G}_{\Omega_i}(k) : L(\widehat{\lambda}_i + p^r \lambda_i)]_G$ is one. We apply Lemma 3.4 and conclude that $\mathcal{G}_{\Omega_i}(k)/\mathcal{G}_{\Omega_{i-1}}(k)$ also embeds in $I_{\widehat{\Gamma}}(\widehat{\lambda}_i) \otimes I_{\Gamma}(\lambda_i)^{(r)}$. The module $I_{\widehat{\Gamma}}(\widehat{\lambda}_i) \otimes I_{\Gamma}(\lambda_i)^{(r)}$ has a filtration with modules of the form $H^0(\widehat{\gamma}_0) \otimes H^0(\gamma_1)^{(r)}$ such that $\lambda_i \uparrow \gamma_0, \gamma_1$. Each of these factors has simple socle with highest weight $\widehat{\gamma}_0 + p^r \gamma_1 \geq (p^r - w_0\sigma)\lambda_i$. The only such weight in Ω_i is $(p^r - w_0\sigma)\lambda_i$. It follows that

$$(4.7.1) \quad \mathcal{G}_{\Omega_i}(k)/\mathcal{G}_{\Omega_{i-1}}(k) \hookrightarrow H^0(\widehat{\lambda}_i) \otimes H^0(\lambda_i)^{(r)}.$$

From the induction hypothesis we conclude that all composition factors of $\mathcal{G}_{\Omega_i}(k)$ are of the form $L(\widehat{\gamma}_0 + p^r \gamma_1)$ with $\gamma_0, \gamma_1 \in \Gamma_i$.

For the remainder of the proof we set $P = P_{\widehat{\Gamma}_i}(\widehat{\gamma}_0) \otimes P_{\Gamma_i}(\gamma_1)^{(r)}$. The module P has a filtration

$$P = P_m \supseteq P_{m-1} \supseteq \cdots \supseteq P_1 \supseteq P_0 = 0$$

with factors of the form $V(\widehat{\nu}_0) \otimes V(\nu_1)^{(r)}$. Consider the LHS spectral sequence:

$$\begin{aligned} E_2^{i,j} &= \text{Ext}_{G/G_r}^i(V(\nu_1)^{(r)}, \text{Ext}_{G_r}^j(V(\widehat{\nu}_0), V(\widehat{\mu}_0)) \otimes V(\mu_1)^{(r)}) \Rightarrow \\ &\text{Ext}_G^{i+j}(V(\widehat{\nu}_0) \otimes V(\nu_1)^{(r)}, V(\widehat{\mu}_0) \otimes V(\mu_1)^{(r)}). \end{aligned}$$

Note that $E_2^{1,0} = 0$ unless $\nu_0 \leq \mu_0$, and from Lemma 3.2, $E_2^{0,1} = 0$ unless $\nu_1 \leq \mu_1$. Hence

$$\text{Ext}_G^1(V(\widehat{\nu}_0) \otimes V(\nu_1)^{(1)}, V(\widehat{\mu}_0) \otimes V(\mu_1)^{(r)}) = 0 \text{ unless } \nu_0 \leq \mu_0 \text{ and } \nu_1 \leq \mu_1.$$

We can rearrange the above filtration such that, for a certain $1 \leq l \leq m$, $P_l \in \text{Mod}(\Omega_i)$ while the kernel S of the projection $P \twoheadrightarrow P_l$ has all its factors $V(\widehat{\nu}_0) \otimes V(\nu_1)^{(r)}$ outside of $\text{Mod}(\Omega_i)$. Observe that

$$\begin{aligned} \text{Hom}_G(V(\widehat{\nu}_0) \otimes V(\nu_1)^{(r)}, \text{ind}_{G_\sigma(\mathbb{F}_q)}^G k) &\cong \text{Hom}_{G_\sigma(\mathbb{F}_q)}(V(\widehat{\nu}_0) \otimes V(\nu_1)^{(r)}, k) \\ &\cong \text{Hom}_{G_\sigma(\mathbb{F}_q)}(V(\widehat{\nu}_0) \otimes V(\sigma\nu_1), k) \\ &\cong \text{Hom}_{G_\sigma(\mathbb{F}_q)}(V(\widehat{\nu}_0), H^0(\widehat{\nu}_1)) \\ &\cong \text{Hom}_G(V(\widehat{\nu}_0), H^0(\widehat{\nu}_1)) = 0, \end{aligned}$$

unless $\nu_0 = \nu_1 \in \Gamma_i$. This implies that $\widehat{\nu}_0 + p^r \nu_1 = (p^r - w_0\sigma)\nu_1 \in \Omega_i$. One concludes that $\text{Hom}_G(S, \mathcal{G}_{\Omega_i}(k)) \hookrightarrow \text{Hom}_G(S, \text{ind}_{G_\sigma(\mathbb{F}_q)}^G k) = 0$. From the two long exact sequences for $\text{Hom}_G(-, \mathcal{G}_{\Omega_i}(k))$ and $\text{Hom}_G(-, \text{ind}_{G_\sigma(\mathbb{F}_q)}^G k)$ associated to $0 \rightarrow S \rightarrow P \rightarrow P_l \rightarrow 0$, this forces

$$(4.7.2) \quad \text{Hom}_G(P, \mathcal{G}_{\Omega_i}(k)) \cong \text{Hom}_G(P_l, \mathcal{G}_{\Omega_i}(k))$$

$$(4.7.3) \quad \cong \text{Hom}_G(P_l, \text{ind}_{G_\sigma(\mathbb{F}_q)}^G k) \cong \text{Hom}_G(P, \text{ind}_{G_\sigma(\mathbb{F}_q)}^G k).$$

Let $L(\widehat{\gamma}_0 + p^r \gamma_1)$ be an arbitrary composition factor of $\mathcal{G}_{\Omega_i}(k)$. Recall from above that $\gamma_0, \gamma_1 \in \Gamma_i$ and $\gamma_0 - \gamma_1$ is in the root lattice. The preceding observations

allow us to argue as follows:

$$\begin{aligned}
[\mathcal{G}_{\Omega_i}(k) : L(\widehat{\gamma}_0 + p^r \gamma_1)]_G & \\
&= \dim \operatorname{Hom}_G(P, \mathcal{G}_{\Omega_i}(k)) \text{ (by Lemma 3.4)} \\
&= \dim \operatorname{Hom}_G(P, \operatorname{ind}_{G_\sigma(\mathbb{F}_q)}^G(k)) \text{ (by 4.7.2-4.7.3)} \\
&= \dim \operatorname{Hom}_{G_\sigma(\mathbb{F}_q)}(P, k) \text{ (by adjointness)} \\
&= \dim \operatorname{Hom}_{G_\sigma(\mathbb{F}_q)}(P_{\widehat{\Gamma}_i}(\widehat{\gamma}_0), I_{\widehat{\Gamma}_i}(\widehat{\gamma}_1)) \text{ (by duality and 2.3.4)} \\
&= \dim \operatorname{Hom}_G(P_{\widehat{\Gamma}_i}(\widehat{\gamma}_0), I_{\widehat{\Gamma}_i}(\widehat{\gamma}_1)) \text{ (by [CPSvdK, Thm 7.4])} \\
&= [I_{\Gamma_i}(\gamma_1) : L(\gamma_0)]_G \text{ (by 2.3.3)} \\
&= \sum_{\nu \in \Gamma_i} ([I_{\Gamma_i}(\gamma_1) : H^0(\nu)]_G \cdot [H^0(\nu) : L(\gamma_0)]_G) \\
&= \sum_{\nu \in \Gamma_i} ([H^0(\nu) : L(\gamma_1)]_G \cdot [H^0(\nu) : L(\gamma_0)]_G) \text{ (see Section 2.3)} \\
&= \sum_{\nu \in \Gamma_i} ([H^0(\nu)^{(r)} : L(\gamma_1)^{(r)}]_G \cdot [H^0(\widehat{\nu}) : L(\widehat{\gamma}_0)]_G) \\
&= \sum_{\nu \in \Gamma_i} [H^0(\widehat{\nu}) \otimes H^0(\nu)^{(r)} : L(\widehat{\gamma}_0 + p^r \gamma_1)]_G.
\end{aligned}$$

For the last equality one makes use of the fact that all composition factors of $H^0(\widehat{\nu})$ are p^r -restricted. One concludes that

$$\operatorname{ch} \mathcal{G}_{\Omega_i}(k) = \sum_{\nu \in \Gamma_i} \operatorname{ch}(H^0(\widehat{\nu}) \otimes H^0(\nu)^{(r)})$$

and

$$\operatorname{ch} \mathcal{G}_{\Omega_i}(k) / \mathcal{G}_{\Omega_{i-1}}(k) = \operatorname{ch}(H^0(\widehat{\lambda}_i) \otimes H^0(\lambda_i)^{(r)}).$$

The embedding (4.7.1) is therefore an isomorphism and the assertion follows. Finally, Proposition 4.4 implies that $\mathcal{G}_{\Omega}(k) = \mathcal{G}_{\Omega_n}(k)$. \square

5. $G_\sigma(\mathbb{F}_q)$ -Extensions

In this section, we apply the filtration of $\mathcal{G}_{\Omega}(k)$ obtained in Theorem 4.7 to make computations about extensions between simple $G_\sigma(\mathbb{F}_q)$ -modules.

5.1. The following theorem involves a minor modification of results proved in [BNP2, Thm. 2.2] and [BNP5, Thm. 2.3]. The proof can be easily adapted to our situation.

THEOREM . *Let $\lambda, \mu \in X_r(T)$. Then*

$$\operatorname{Ext}_{G_\sigma(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) \cong \operatorname{Ext}_G^1(L(\lambda), L(\mu) \otimes \mathcal{G}_{\Omega}(k)).$$

5.2. Theorem 4.7 implies that $\mathcal{G}_{\Omega}(k)$ has a filtration with factors of the form $H^0(\widehat{\nu}) \otimes H^0(\nu)^{(r)}$ for $\nu \in \Gamma$. In order to apply Theorem 5.1 and obtain information about $\operatorname{Ext}_{G_\sigma(\mathbb{F}_q)}^1(L(\lambda), L(\mu))$, we investigate the Ext-groups

$$\begin{aligned}
\operatorname{Ext}_G^1(L(\lambda), L(\mu) \otimes H^0(\widehat{\eta}) \otimes H^0(\eta)^{(r)}) &\cong \operatorname{Ext}_G^1(L(\lambda) \otimes V(-w_0\eta)^{(r)}, L(\mu) \otimes H^0(\widehat{\eta})) \\
&\cong \operatorname{Ext}_G^1(L(\lambda) \otimes V(\nu)^{(r)}, L(\mu) \otimes H^0(\sigma\nu)),
\end{aligned}$$

where $\nu = -w_0\eta$. The following lemma says that for these groups to be non-zero, we must in fact have $\nu \in \Gamma_h = \{\nu \in X(T)^+ \mid \langle \nu, \alpha_0^\vee \rangle < h\}$.

LEMMA . Let $\lambda, \mu \in X_r(T)$ and $\nu \in X(T)_+$ with $\langle \nu, \alpha_0^\vee \rangle < p^r$. If $p = 2$, assume further that $p^r > 4$. If $\text{Ext}_G^1(L(\lambda) \otimes V(\nu)^{(r)}, L(\mu) \otimes H^0(\sigma\nu)) \neq 0$, then $\langle \nu, \alpha_0^\vee \rangle \leq (h-1)$. (In the excluded cases for $p = 2$, one can replace $h-1$ by h .)

PROOF. Consider the LHS spectral sequence

$$\begin{aligned} E_2^{i,j} &= \text{Ext}_{G/G_r}^i(V(\nu)^{(r)}, \text{Ext}_{G_r}^j(L(\lambda), L(\mu) \otimes H^0(\sigma\nu))) \\ &\Rightarrow \text{Ext}_G^{i+j}(L(\lambda) \otimes V(\nu)^{(r)}, L(\mu) \otimes H^0(\sigma\nu)). \end{aligned}$$

Notice that, by Proposition 3.1,

$$E_2^{1,0} = \text{Ext}_{G/G_r}^1(V(\nu)^{(r)}, k) \otimes \text{Hom}_G(L(\lambda), L(\mu) \otimes H^0(\sigma\nu)).$$

Hence, $E_2^{1,0} = 0$ by [Jan1, II.4.13] and so

$$E_2 \cong E_2^{0,1} \cong \text{Hom}_{G/G_r}(V(\nu)^{(r)}, \text{Ext}_{G_r}(L(\lambda), L(\mu) \otimes H^0(\sigma\nu))).$$

Let $p^r\gamma$ be a weight of $\text{Ext}_{G_r}^1(L(\lambda), L(\mu) \otimes H^0(\sigma\nu))$. It follows from the argument in [BNP4, 5.2]) that

$$(5.2.1) \quad p^r \langle \gamma, \alpha_0^\vee \rangle \leq \langle \lambda, \alpha_0^\vee \rangle + \langle \mu, \alpha_0^\vee \rangle + \langle \sigma\nu, \alpha_0^\vee \rangle + 3p^{r-1}.$$

Consider the short exact sequence $0 \rightarrow L(\mu) \rightarrow \text{St}_r \otimes L((p^r-1)\rho + w_0\mu) \rightarrow R \rightarrow 0$. Using the long exact sequence in cohomology and the fact that St_r is injective as a G_r -module one obtains a surjection

$$\text{Hom}_{G_r}(L(\lambda), R \otimes H^0(\sigma\nu)) \twoheadrightarrow \text{Ext}_{G_r}^1(L(\lambda), L(\mu) \otimes H^0(\sigma\nu)).$$

Hence, any weight $p^r\gamma$ of $\text{Ext}_{G_r}^1(L(\lambda), L(\mu) \otimes H^0(\sigma\nu))$ also satisfies

$$(5.2.2) \quad p^r \langle \gamma, \alpha_0^\vee \rangle \leq 2(p^r-1)(h-1) - \langle \lambda, \alpha_0^\vee \rangle - \langle \mu, \alpha_0^\vee \rangle + \langle \sigma\nu, \alpha_0^\vee \rangle.$$

Adding equations (5.2.1) and (5.2.2) and dividing by two yields

$$(5.2.3) \quad p^r \langle \gamma, \alpha_0^\vee \rangle \leq (p^r-1)(h-1) + \langle \sigma\nu, \alpha_0^\vee \rangle + \frac{3}{2}p^{r-1}$$

$$(5.2.4) \quad = \langle \sigma\nu, \alpha_0^\vee \rangle + (p^r-1)h + 1 - p^r \left(1 - \frac{3}{2p}\right).$$

Replacing γ by ν results in

$$(p^r-1)\langle \nu, \alpha_0^\vee \rangle \leq (p^r-1)h + 1 - p^r \left(1 - \frac{3}{2p}\right) < (p^r-1)h$$

and the assertion follows. \square

5.3. We can now use Theorem 5.1 to show that $\text{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu))$ embeds as a submodule of a certain direct sum of G -extensions. The conclusion is not as strong as in [BNP2, 2.5.3.2] (where equality was shown to hold), however the assumption on the prime is much less restrictive here. Moreover, this result is sufficient to obtain several nice applications which will follow.

Let M be a G -module with a filtration $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_l = M$. Then one can argue inductively that for all $N \in \text{Mod}(G)$ and $i \geq 0$

$$(5.3.1) \quad \dim \text{Ext}_G^i(N, M) \leq \sum_{i=1}^l \dim \text{Ext}_G^i(N, M_i/M_{i-1}).$$

PROPOSITION . Let $p^{r-1}(p-1) > 4h-6$, $\lambda, \mu \in X_r(T)$, and $\Gamma'_h = \Gamma_h - \{0\}$.
(a) Assume $r \geq 1$, then

$$\mathrm{Ext}_{G_\sigma(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) \hookrightarrow \mathrm{Ext}_G^1(L(\lambda), L(\mu)) \oplus R,$$

where

$$\begin{aligned} R &\cong \bigoplus_{\nu \in \Gamma'_h} \mathrm{Ext}_G^1(L(\lambda) \otimes V(\nu)^{(r)}, L(\mu) \otimes H^0(\sigma\nu)). \\ &\cong \bigoplus_{\nu \in \Gamma'_h} \mathrm{Hom}_{G/G_r}(V(\nu)^{(r)}, \mathrm{Ext}_{G_r}^1(L(\lambda), L(\mu) \otimes H^0(\sigma\nu))). \end{aligned}$$

(b) Assume $r \geq 2$ and let $s = \lfloor \frac{r}{2} \rfloor$. Assume further that $p^s \geq h$. Set $\lambda = \lambda_0 + p^s \lambda_1$ and $\mu = \mu_0 + p^s \mu_1$ with $\lambda_0, \mu_0 \in X_s(T)$ and $\lambda_1, \mu_1 \in X_{r-s}(T)$. Then we may reidentify R as

$$\begin{aligned} &\bigoplus_{\nu \in \Gamma'_h} (\mathrm{Ext}_G^1(L(\lambda_1) \otimes V(\nu)^{(r-s)}, L(\mu_1)) \otimes \mathrm{Hom}_G(L(\lambda_0), L(\mu_0) \otimes H^0(\sigma\nu))) \cong \\ &\bigoplus_{\nu \in \Gamma'_h} (\mathrm{Hom}_G(V(\nu)^{(r-s)}, \mathrm{Ext}_{G_{r-s}}^1(L(\lambda_1), L(\mu_1))) \otimes \mathrm{Hom}_G(L(\lambda_0), L(\mu_0) \otimes H^0(\sigma\nu))). \end{aligned}$$

PROOF. From Theorem 4.7, $\mathcal{G}_\Omega(k)$ has a filtration with factors of the form $H^0(\hat{\nu}) \otimes H^0(\nu)^{(r)}$ where each $\nu \in \Gamma$ appears exactly once. By Theorem 5.1, Lemma 5.2 (and the remarks), and (5.3.1), we have

$$\begin{aligned} \dim \mathrm{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) &= \dim \mathrm{Ext}_G^1(L(\lambda), L(\mu) \otimes \mathcal{G}_\Omega(k)) \\ &\leq \sum_{\nu \in \Gamma_h} \dim \mathrm{Ext}_G^1(L(\lambda) \otimes V(\nu)^{(r)}, L(\mu) \otimes H^0(\sigma\nu)) \\ &= \dim \mathrm{Ext}_G^1(L(\lambda), L(\mu)) \\ &\quad + \sum_{\nu \in \Gamma_h - \{0\}} \dim \mathrm{Ext}_G^1(L(\lambda) \otimes V(\nu)^{(r)}, L(\mu) \otimes H^0(\sigma\nu)). \end{aligned}$$

Since all modules involved are trivial as G -modules, the claimed embedding follows.

Consider the LHS spectral sequence

$$\begin{aligned} E_2^{i,j} &= \mathrm{Ext}_{G/G_r}^i(V(\nu)^{(r)}, \mathrm{Ext}_{G_r}^j(L(\lambda), L(\mu) \otimes H^0(\sigma\nu))) \\ &\Rightarrow \mathrm{Ext}_G^{i+j}(L(\lambda) \otimes V(\nu)^{(r)}, L(\mu) \otimes H^0(\sigma\nu)). \end{aligned}$$

As in the proof of Lemma 5.2, it follows that $E_2^{1,0} = 0$ and hence $E^2 \cong E_2^{0,1}$, which gives part (a).

Now assume $r \geq 2$. Without loss of generality (by dualizing if necessary), we may assume that $\langle \mu_1, \alpha_0^\vee \rangle \leq \langle \lambda_1, \alpha_0^\vee \rangle$. We use the LHS spectral sequence

$$\begin{aligned} E_2^{i,j} &= \mathrm{Ext}_{G/G_s}^i(L(\lambda_1)^{(s)} \otimes V(\nu)^{(r)}, \mathrm{Ext}_{G_s}^j(L(\lambda_0), L(\mu_0) \otimes H^0(\sigma\nu))) \otimes L(\mu_1)^{(s)} \\ &\Rightarrow \mathrm{Ext}_G^{i+j}(L(\lambda) \otimes V(\nu)^{(r)}, L(\mu) \otimes H^0(\sigma\nu)). \end{aligned}$$

It follows from (5.2.4) that any weight $p^s \gamma$ of $\mathrm{Ext}_{G_s}^1(L(\lambda_0), L(\mu_0) \otimes H^0(\sigma\nu))$ satisfies $p^s \langle \gamma, \alpha_0^\vee \rangle < (p^s - 1)h + 1 - \frac{p^s}{4} + \langle \nu, \alpha_0^\vee \rangle \leq p^r - \frac{5}{4}p^s + 1 + \langle \nu, \alpha_0^\vee \rangle < p^r - 1 + \langle \nu, \alpha_0^\vee \rangle$. On the other hand, if $\nu \neq 0$, the module $L(\lambda_1)^{(s)} \otimes V(\nu)^{(r)}$ has simple head with weight $p^s \lambda_1 + p^r \nu$. Comparison of weights forces $E^{0,1} = 0$.

Therefore, by Proposition 3.1,

$$\begin{aligned} \text{Ext}_G^1(L(\lambda) \otimes V(\nu)^{(r)}, L(\mu) \otimes H^0(\sigma\nu)) &\cong E_2^{1,0} \\ &= \text{Ext}_{G/G_s}^1(L(\lambda_1)^{(s)} \otimes V(\nu)^{(r)}, \text{Hom}_{G_s}(L(\lambda_0), L(\mu_0) \otimes H^0(\sigma\nu)) \otimes L(\mu_1)^{(s)}) \\ &\cong \text{Ext}_{G/G_s}^1(L(\lambda_1)^{(s)} \otimes V(\nu)^{(r)}, L(\mu_1)^{(s)}) \otimes \text{Hom}_G(L(\lambda_0), L(\mu_0) \otimes H^0(\sigma\nu)) \\ &\cong \text{Ext}_G^1(L(\lambda_1) \otimes V(\nu)^{(r-s)}, L(\mu_1)^{(s)}) \otimes \text{Hom}_G(L(\lambda_0), L(\mu_0) \otimes H^0(\sigma\nu)), \end{aligned}$$

which gives the first reidentification.

To investigate $\text{Ext}_G^1(L(\lambda_1) \otimes V(\nu)^{(r-s)}, L(\mu_1))$, consider the LHS spectral sequence

$$\begin{aligned} E_2^{i,j} &= \text{Ext}_{G/G_{r-s}}^i(V(\nu)^{(r-s)}, \text{Ext}_{G_{r-s}}^j(L(\lambda_1), L(\mu_1))) \\ &\Rightarrow \text{Ext}_G^{i+j}(L(\lambda_1) \otimes V(\nu)^{(r-s)}, L(\mu_1)). \end{aligned}$$

As before, using Proposition 3.1 (with $M = k$), we see that $E_2^{1,0} = 0$ which gives the result. \square

REMARK . Note that the condition on the prime given in part (b) is almost always stronger than the initial assumption of the proposition. If r is odd and greater than one, then the assumption that $p^s \geq h$ implies $p^{r-1}(p-1) > 4h-6$. Indeed, we have

$$4h-6 \leq 4h-6 + (h-2)^2 = h^2 - 2 < h^2 \leq (p^s)^2 = p^{r-1} \leq p^{r-1}(p-1).$$

For r even, it is a bit more involved but straightforward exercise to show that the condition $p^s \geq h$ implies $p^{r-1}(p-1) > 4h-6$ as long as $p > 3$. If $p = 3$, the implication fails only when $r = 2$ and $h = 3$. If $p = 2$, the implication fails only when $h = 2, 3, 4, 5$, or 6 .

COROLLARY . Assume $r \geq 2$ and let $s = \lfloor \frac{r}{2} \rfloor$. Assume that $p^s \geq h$ and $p^{r-1}(p-1) > 4h-6$. Given $\lambda, \mu \in X_r(T)$, let $\lambda = \lambda_0 + p^s \lambda_1$ and $\mu = \mu_0 + p^s \mu_1$ with $\lambda_0, \mu_0 \in X_s(T)$ and $\lambda_1, \mu_1 \in X_{r-s}(T)$. If either of the following conditions hold:

- (a) $\text{Ext}_{G_{r-s}}^1(L(\lambda_1), L(\mu_1)) = 0$
- (b) $\text{Hom}_G(L(\lambda_0), L(\mu_0) \otimes H^0(\sigma\nu)) = 0$,

then $\text{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) \cong \text{Ext}_G^1(L(\lambda), L(\mu))$.

PROOF. If either condition holds, then it follows from part (b) of the proposition that the remainder term R is zero and hence $\text{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu))$ embeds in $\text{Ext}_G^1(L(\lambda), L(\mu))$. On the other hand, from [CPSvdK], we know that the restriction map $\text{Ext}_G^1(L(\lambda), L(\mu)) \rightarrow \text{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu))$ is an embedding. Hence it must be an isomorphism. \square

5.4. Self-extensions for small primes. The following theorem improves on [BNP2, Thm. 3.4] because it can be applied to primes which are smaller than $3(h-1)$. First, we need an observation to deal with the special case of type C_n when $p = 2$.

LEMMA . Let $p = 2$, Φ be of type C_n , and $\lambda \in X_r(T)$. Then $\text{Ext}_{G_r}^1(L(\lambda), L(\lambda))$ is either zero or isomorphic to $N^{(r)}$, where N is a G -module whose weights are not contained in the root lattice.

PROOF. $\text{Ext}_{G_r}^1(L(\lambda), L(\lambda))$ embeds in $\text{Ext}_{G_r}^1(L(\lambda), H^0(\lambda))$. By [Jan1, II.12.8], $\text{Ext}_{G_r}^1(L(\lambda), H^0(\lambda)) \cong \text{ind}_B^G(\text{Ext}_{B_r}^1(L(\lambda), \lambda))$. Assume that $\text{Ext}_{B_r}^1(L(\lambda), \lambda) \neq 0$. It follows from [And1] (see also [Jan1, II.12.1 - 12.5]) that the B -socle of $\text{Ext}_{B_r}^1(L(\lambda), \lambda)$ is isomorphic to $\frac{1}{2}\alpha_n$ where α_n denotes the unique last simple root of Φ . Hence, the weights of $\text{Ext}_{G_r}^1(L(\lambda), L(\lambda))$ are not in the root lattice. \square

THEOREM . Assume $r \geq 2$ and let $s = \lfloor \frac{r}{2} \rfloor$. Assume $p^s \geq h$ and $p^{r-1}(p-1) > 4h - 6$. Then

$$\text{Ext}_{G_\sigma(\mathbb{F}_q)}^1(L(\lambda), L(\lambda)) = 0$$

for all $\lambda \in X_r(T)$.

PROOF. Let $\lambda = \lambda_0 + p^s \lambda_1$ with $\lambda_0 \in X_s(T)$ and $\lambda_1 \in X_{r-s}(T)$. If $p \neq 2$ or Φ is not of type C_n , since $\text{Ext}_{G_{r-s}}^1(L(\lambda_1), L(\lambda_1)) = 0$, it follows from Corollary 5.3 that $\text{Ext}_{G_\sigma(\mathbb{F}_q)}^1(L(\lambda), L(\lambda)) \cong \text{Ext}_G^1(L(\lambda), L(\lambda)) = 0$ as claimed.

If $p = 2$ and Φ is of type C_n , by Proposition 5.3, it suffices to show that for all $\nu \in \Gamma_h - \{0\}$

$$\text{Hom}_G(V(\nu)^{(r-s)}, \text{Ext}_{G_{r-s}}^1(L(\lambda_1), L(\lambda_1))) \otimes \text{Hom}_G(L(\lambda_0), L(\lambda_0) \otimes H^0(\sigma\nu)) = 0.$$

$\text{Hom}_G(L(\lambda_0), L(\lambda_0) \otimes H^0(\sigma\nu)) \neq 0$ implies that $\sigma\nu$ and hence ν are contained in the root lattice. On the other hand, $\text{Hom}_G(V(\nu)^{(r-s)}, \text{Ext}_{G_{r-s}}^1(L(\lambda_1), L(\lambda_1))) \neq 0$ forces ν to be outside the root lattice by the above lemma. \square

5.5. Cohomology for small primes. The following shows that the first $G_\sigma(\mathbb{F}_q)$ -cohomology with coefficients in a simple module can be described in terms of the first G -cohomology with coefficients in a simple p^r -restricted module, provided r is sufficiently large. No condition on the prime is necessary.

THEOREM . Assume $r \geq 2$ and let $s = \lfloor \frac{r}{2} \rfloor$. Assume $p^{s-1}(p-1) > h$. Given $\lambda \in X_r(T)$, let $\lambda = \lambda_0 + p^s \lambda_1$ with $\lambda_0 \in X_s(T)$ and $\lambda_1 \in X_{r-s}(T)$. Define $\tilde{\lambda} = \sigma\lambda_1 + p^{r-s}\lambda_0$. Then

$$\text{H}^1(G_\sigma(\mathbb{F}_q), L(\lambda)) \cong \begin{cases} \text{H}^1(G, L(\tilde{\lambda})) & \text{if } \lambda_0 \in \Gamma_h - \{0\} \\ \text{H}^1(G, L(\lambda)) & \text{else.} \end{cases}$$

PROOF. Note that if $p^{s-1}(p-1) > h$, then $p^s > h$ and $p^{r-1}(p-1) > 4h - 6$ so the results in 5.3 may be applied. The Frobenius map is an automorphism on $G_\sigma(\mathbb{F}_q)$ and $L(\lambda)^{(r-s)} \cong L(\tilde{\lambda})$ as a $G_\sigma(\mathbb{F}_q)$ -module. Therefore, $\text{H}^1(G_\sigma(\mathbb{F}_q), L(\lambda)) \cong \text{H}^1(G_\sigma(\mathbb{F}_q), L(\lambda)^{(r-s)}) \cong \text{H}^1(G_\sigma(\mathbb{F}_q), L(\tilde{\lambda}))$. Since

$$\text{H}^1(G, L(\lambda)) \hookrightarrow \text{H}^1(G_\sigma(\mathbb{F}_q), L(\lambda)) \text{ and } \text{H}^1(G, L(\tilde{\lambda})) \hookrightarrow \text{H}^1(G_\sigma(\mathbb{F}_q), L(\tilde{\lambda})),$$

one concludes that $\text{H}^1(G_\sigma(\mathbb{F}_q), L(\lambda)) = 0$ implies $\text{H}^1(G, L(\lambda)) = \text{H}^1(G, L(\tilde{\lambda})) = 0$.

Assume that $\text{H}^1(G, L(\tilde{\lambda})) \neq 0$. If $\lambda_0 \notin \Gamma_h - \{0\}$, then $\text{Hom}_G(k, L(\lambda_0) \otimes H^0(\sigma\nu)) \cong \text{Hom}_G(V(\hat{\nu}), L(\lambda_0)) = 0$, for all $\nu \in \Gamma_h - \{0\}$. Corollary 5.3 (part (b)) now implies that

$$\text{H}^1(G_\sigma(\mathbb{F}_q), L(\lambda)) \cong \text{H}^1(G, L(\lambda)).$$

If $\lambda_0 \in \Gamma_h - \{0\}$, then we apply Proposition 5.3 to $\text{H}^1(G_\sigma(\mathbb{F}_q), L(\tilde{\lambda}))$. Notice that $r-s \geq s$ and the same argument as in the proof of Proposition 5.3 yields that

$$\text{H}^1(G_\sigma(\mathbb{F}_q), L(\tilde{\lambda})) \hookrightarrow \text{H}^1(G, L(\tilde{\lambda})) \oplus R$$

where

$$R = \bigoplus_{\nu \in \Gamma_h - \{0\}} \text{Hom}_G(V(\nu)^{(s)}, H^1(G_s, L(\lambda_0))) \otimes \text{Hom}_G(k, L(\sigma\lambda_1) \otimes H^0(\sigma\nu)).$$

It follows from Lemma 3.2 that $H^1(G_s, L(\lambda_0)) \cong H^1(G, L(\lambda_0))$. Hence,

$$\text{Hom}_G(V(\nu)^{(s)}, H^1(G_s, L(\lambda_0))) = 0$$

for all $\nu \in \Gamma_h - \{0\}$ and $H^1(G_\sigma(\mathbb{F}_q), L(\tilde{\lambda})) \cong H^1(G, L(\tilde{\lambda}))$, as claimed. \square

5.6. Extensions between simple modules for small primes. The following theorem generalizes [BNP5, Thm 3.2(a)] to arbitrary primes but large prime powers.

THEOREM . *Assume $r \geq 3$ and let $s = \lceil \frac{r-1}{2} \rceil$. Assume $p^s \geq h$. Given $\lambda, \mu \in X_r(T)$ such that let $\lambda = \sum_{i=0}^{r-1} p^i \lambda_i$ and $\mu = \sum_{i=0}^{r-1} p^i \mu_i$ with $\lambda_i, \mu_i \in X_1(T)$. Then there exists an integer $0 \leq n < r$ such that*

$$\text{Ext}_{G_\sigma(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) \cong \text{Ext}_G^1(L(\tilde{\lambda}), L(\tilde{\mu})),$$

where

$$\begin{aligned} \tilde{\lambda} &= \sum_{i=0}^{n-1} p^i \sigma(\lambda_{i+r-n}) + \sum_{i=n}^{r-1} p^i \lambda_{i-n} \in X_r(T), \\ \tilde{\mu} &= \sum_{i=0}^{n-1} p^i \sigma(\mu_{i+r-n}) + \sum_{i=n}^{r-1} p^i \mu_{i-n} \in X_r(T). \end{aligned}$$

PROOF. Note that that $p^s \geq h$ implies $p^{r-1}(p-1) > 4h-6$. Indeed, we have $4h-6 \leq 4h-6 + (h-2)^2 = h^2 - 2 < h^2 \leq (p^s)^2 = p^{2s} \leq p^{r-1} \leq p^{r-1}(p-1)$.

Thus the results in 5.3 and 5.4 may be applied. If $\lambda = \mu$ the claim follows from Theorem 5.4. Assume $\lambda \neq \mu$. Then there exists $0 \leq l < r$ with $\lambda_l \neq \mu_l$. If $l \leq s$ we set $n = s - l$ and for $l \geq s$ we set $n = r + s - l$. As before we note that $L(\lambda)^{(n)} \cong L(\tilde{\lambda})$ and $L(\mu)^{(n)} \cong L(\tilde{\mu})$ as $G_\sigma(\mathbb{F}_q)$ -modules. Therefore, $\text{Ext}_{G_\sigma(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) \cong \text{Ext}_{G_\sigma(\mathbb{F}_q)}^1(L(\lambda)^{(n)}, L(\mu)^{(n)}) \cong \text{Ext}_{G_\sigma(\mathbb{F}_q)}^1(L(\tilde{\lambda}), L(\tilde{\mu}))$. Moreover $\tilde{\lambda}_s = \lambda_l \neq \mu_l = \tilde{\mu}_s$.

Set $\lambda' = \sum_{i=0}^{s-1} p^i \tilde{\lambda}_i$, $\lambda'' = \sum_{i=s+1}^{r-1} p^{i-s-1} \tilde{\lambda}_i$, and $\mu' = \sum_{i=0}^{s-1} p^i \tilde{\mu}_i$, $\mu'' = \sum_{i=s+1}^{r-1} p^{i-s-1} \tilde{\mu}_i$. Then

$$\tilde{\lambda} = \lambda' + p^s \lambda_l + p^{s+1} \lambda'' \text{ and } \tilde{\mu} = \mu' + p^s \mu_l + p^{s+1} \mu''.$$

Since $s = \lceil \frac{r-1}{2} \rceil$ implies $r - s - 1 \geq s$, we can use the same arguments as in 5.3 to conclude that $\text{Ext}_{G_\sigma(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) \hookrightarrow \text{Ext}_G^1(L(\tilde{\lambda}), L(\tilde{\mu})) \oplus R$, where R is isomorphic to

$$\bigoplus_{\nu \in \Gamma'_h} (\text{Ext}_G^1(L(\lambda'') \otimes V(\nu)^{(r-s-1)}, L(\mu'')) \otimes \text{Hom}_G(L(\lambda' + p^s \lambda_l), L(\mu' + p^s \mu_l) \otimes H^0(\sigma\nu))$$

with $\Gamma'_h = \Gamma_h - \{0\}$. From Proposition 3.1, one obtains

$$\begin{aligned} &\text{Hom}_G(L(\lambda' + p^s \lambda_l), L(\mu' + p^s \mu_l) \otimes H^0(\sigma\nu)) \\ &\cong \text{Hom}_{G/G_s}(L(\lambda_l)^{(s)}, \text{Hom}_{G_s}(L(\lambda'), L(\mu') \otimes H^0(\sigma\nu)) \otimes L(\mu_l)^{(s)}) \\ &\cong \text{Hom}_G(L(\lambda_l), L(\mu_l)) \otimes \text{Hom}_G(L(\lambda'), L(\mu') \otimes H^0(\sigma\nu)). \end{aligned}$$

Now $\lambda_l \neq \mu_l$ forces $R = 0$ and the assertion follows. \square

As a corollary we can show that for sufficiently large r the dimension of $G_\sigma(\mathbb{F}_q)$ -extensions between simple modules is bounded by the dimension of G -extensions between simple p^r -restricted modules.

COROLLARY . Assume $r \geq 3$ and let $s = \lceil \frac{r-1}{2} \rceil$. Assume $p^s \geq h$. Then

$$\begin{aligned} \max\{\dim_k \text{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) \mid \lambda, \mu \in X_r(T)\} \\ = \max\{\dim_k \text{Ext}_G^1(L(\lambda), L(\mu)) \mid \lambda, \mu \in X_r(T)\}. \end{aligned}$$

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