

EXTENSIONS FOR FINITE CHEVALLEY GROUPS I.

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1 Introduction

1.1 Let G be a connected semisimple algebraic group defined and split over the field \mathbb{F}_p with p elements, and k be the algebraic closure of \mathbb{F}_p . Assume further that G is almost simple and simply connected. Moreover, let $G(\mathbb{F}_q)$ be the finite Chevalley group consisting of \mathbb{F}_q -rational points of G where $q = p^r$ for a non-negative integer r . Let G_r denote the r th Frobenius kernel of G . For any rational G -module V , let $V^{(l)}$ denote the rational G -module obtained by l twists by the Frobenius endomorphism. In 1977, Cline, Parshall, Scott and van der Kallen [CPSK] proved an important stability result which relates the rational cohomology of G with the cohomology of the finite group $G(\mathbb{F}_q)$. More precisely, they show that for any fixed non-negative integer n , if V is a finite dimensional rational G -module, then $H^n(G, V^{(l)}) \cong H^n(G(\mathbb{F}_q), V^{(l)}) \cong H^n(G(\mathbb{F}_q), V)$ for sufficiently large r and l . In other words, the restriction map $H^n(G, V^{(l)}) \rightarrow H^n(G(\mathbb{F}_q), V^{(l)})$ is an isomorphism for large r and l .

The purpose of this paper is to give explicit formulas for the extensions between simple modules of finite Chevalley groups. A complete explanation for the notation to follow can be found in Section 1.2. It is well-known that the simple G -modules are indexed in a natural way by the set of dominant weights $X(T)_+$. For each $\lambda \in X(T)_+$, let $L(\lambda)$ be the corresponding simple module. Let $X_r(T)$ be the set of p^r -restricted weights in $X(T)_+$. For $\lambda \in X_r(T)$, $L(\lambda)$ remains simple upon restriction to $G(\mathbb{F}_q)$ and G_r . In fact a complete set of simple modules for $G(\mathbb{F}_q)$ and G_r are obtained in this way.

We will now describe the main results in this paper. In Section 2, Theorem 2.2 gives a formula relating extensions between simple $G(\mathbb{F}_q)$ -modules and extensions over G . The formula holds for arbitrary primes but involves a certain truncated induced module $\mathcal{G}(k)$. The module $\mathcal{G}(k)$ was introduced in previous work of the authors [BNP1] that relates extensions between $\text{Mod}(G(\mathbb{F}_q))$, $\text{Mod}(G)$, and $\text{Mod}(G_r)$ via spectral sequences. Here $\text{Mod}(G(\mathbb{F}_q))$ denotes the category of $kG(\mathbb{F}_q)$ -modules, $\text{Mod}(G)$ denotes the category of rational G -modules, and $\text{Mod}(G_r)$ denotes the category of rational G_r -modules.

In general the structure of $\mathcal{G}(k)$ is rather mysterious. However, for $p \geq 3(h-1)$, where h is the Coxeter number associated with the root system for G , the module $\mathcal{G}(k)$ is semisimple and this leads to a more computationally useful Ext^1 -formula given in Theorem 2.5.

Date: January 2003.

1991 *Mathematics Subject Classification.* Primary 20C, 20G; Secondary 20J06, 20G10.

Research of the first author was supported in part by NSA grant MDA904-02-1-0078.

Research of the second author was supported in part by NSF grant DMS-0102225.

For $p \geq 3(h-1)$ and $\lambda, \mu \in X_r(T)$,

$$\mathrm{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) \cong \bigoplus_{\nu \in \Gamma} \mathrm{Ext}_G^1(L(\lambda) \otimes L(\nu)^{(r)}, L(\mu) \otimes L(\nu))$$

where $\Gamma = \{\nu \in X(T)_+ \mid \langle \nu, \alpha_0^\vee \rangle \leq h-1\}$.

We have recently applied this formula to obtain rather explicit information about the cohomologies for finite Chevalley groups and their relationship to the cohomologies of the ambient algebraic groups and their Frobenius kernels in [BNP2].

In this paper, the main application of this formula is to answer long standing questions involving the existence of self-extensions of simple modules for finite Chevalley groups. In 1985, J.E. Humphreys [Hum] speculated about the existence of such self-extensions for $G(\mathbb{F}_q)$. In particular, he showed that for $r=1$ and for the root system of type C_2 , there is an interesting family of self-extensions. Theorem 3.4 says that there are no self-extensions for simple $G(\mathbb{F}_q)$ -modules for $r \geq 2$ and $p \geq 3(h-1)$. That is, $\mathrm{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\lambda)) = 0$ for all $\lambda \in X_r(T)$ with $r \geq 2$ and $p \geq 3(h-1)$.

The determination of self-extensions is much more subtle in the $r=1$ case. The bulk of Section 4 is devoted to proving Theorem 4.2 which says that for $\lambda \in X_1(T)$, $\mathrm{Ext}_{G(\mathbb{F}_p)}^1(L(\lambda), L(\lambda)) = 0$ with a few exceptions on λ when the root system is of type A_1 or C_n . Our results provide answers to many of the questions raised in Humphreys' paper. In fact, our proofs clearly demonstrate how his family of self-extensions naturally arises. The methods used in this paper are based on ideas of H.H. Andersen [And1], who earlier proved that self-extensions occur for simple G_r -modules only if $p=2$ and the underlying root system is of type A_1 or C_n .

Recent work by Tiep and Zalesskii [TZ, Proposition 1.4] provides an important application of self-extensions. Specifically, they have shown that certain simple modular representations can be lifted to characteristic zero only if they admit self-extensions. Moreover, they found a new class of simple modules for groups of type C_n with $n > 2$ that admit self-extensions. These examples of self-extensions were discovered independently in [P3]. It remains an open problem to completely classify all simple modules that admit self-extensions (even for large primes).

In the final section of the paper, the self-extensions results are used to obtain criteria for the semisimplicity of finite-dimensional $G(\mathbb{F}_q)$ -modules.

1.2 Notation: Throughout this paper G is a connected semisimple algebraic group defined and split over the finite field \mathbb{F}_p with p elements. Assume further that G is almost simple and simply connected. The results in this paper have immediate generalizations to the semisimple case. The field k is the algebraic closure of \mathbb{F}_p and G will be considered as an algebraic group scheme over k . Let Φ be a root system associated to G with respect to a maximal split torus T . Moreover, let Φ^+ (resp. Φ^-) be positive (resp. negative) roots and Δ be a base consisting of simple roots. Let B be a Borel subgroup containing T corresponding to the negative roots and U be the unipotent radical of B .

The Euclidean space associated with Φ will be denoted by \mathbb{E} and the inner product on \mathbb{E} will be denoted by $\langle \cdot, \cdot \rangle$. Let $X(T)$ be the integral weight lattice obtained from Φ . The set $X(T)$ has a partial ordering defined as follows. If $\lambda, \mu \in X(T)$ then $\lambda \geq \mu$ if and only if $\lambda - \mu \in \sum_{\alpha \in \Delta} \mathbb{N}\alpha$. Let $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ be the coroot corresponding to $\alpha \in \Phi$ and α_0^\vee

denote the coroot with α_0 being the highest short root. Moreover, let ρ be the half sum of positive roots and w_0 denote the long element of the Weyl group. The Coxeter number associated to Φ is $h = \langle \rho, \alpha_0^\vee \rangle + 1$. The set of dominant integral weights is defined by

$$X(T)_+ = \{\lambda \in X(T) : 0 \leq \langle \lambda, \alpha^\vee \rangle \text{ for all } \alpha \in \Delta\}.$$

Furthermore, the set of p^r -restricted weights is

$$X_r(T) = \{\lambda \in X(T) : 0 \leq \langle \lambda, \alpha^\vee \rangle < p^r \text{ for all } \alpha \in \Delta\}.$$

For any $\lambda \in X(T)$, let $H^j(\lambda) = R^j \text{ind}_B^G \lambda$ for $j \geq 0$. The simple modules for G are labelled by the set $X(T)_+$ and denoted by $L(\lambda)$, $\lambda \in X(T)_+$ with $L(\lambda) = \text{soc}_G H^0(\lambda)$. A complete set of non-isomorphic simple G_r -modules and simple $G(\mathbb{F}_q)$ -modules are obtained by taking $\{L(\lambda) : \lambda \in X_r(T)\}$.

For a vector space V over k , the dual space will be denoted $V^* = \text{Hom}(V, k)$. Note that for a dominant weight λ , $L(\lambda)^* \cong L(-w_0\lambda)$. The Weyl module $V(\lambda)$ is the dual module of $H^0(-w_0\lambda)$. Let H denote either G , $G(\mathbb{F}_q)$, or G_r and M, N, Q be finite-dimensional H -modules. In numerous places in the paper, we will use the fact that $\text{Ext}_H^i(M, N \otimes Q) \cong \text{Ext}_H^i(M \otimes Q^*, N)$ for all $i \geq 0$ (cf. [Jan1, I 4.4]). In conjunction with this isomorphism, we will also use the facts that $(M^*)^* \cong M$ and $\langle \lambda, \alpha_0^\vee \rangle = \langle -w_0\lambda, \alpha_0^\vee \rangle$ for $\lambda \in X(T)$.

1.3 Acknowledgments: The first author and the third author thank the Department of Mathematics and Statistics at Utah State University for its hospitality during the preparation of this paper. The third author would like to thank the Mathematics Department of the University of Oregon where the author spent his sabbatical while parts of this paper were written. The authors would also like to acknowledge the referee for several useful suggestions.

2 General Ext¹-formula

In this section we give two formulas for Ext^1 between simple $G(\mathbb{F}_q)$ -modules in terms of Ext^1 between G -modules. The first, Theorem 2.2, holds for any prime p . The second and more useful formula, Theorem 2.5, holds for $p \geq 3(h-1)$. It is this second formula that will be exploited in Sections 3 and 4 to analyze self-extensions.

2.1 We briefly review the techniques developed in [BNP1]. Let \mathcal{C} be the full subcategory of $\text{Mod}(G)$ with objects having composition factors whose highest weight lies in π where

$$\pi = \{\lambda \in X(T)_+ : \langle \lambda + \rho, \alpha_0^\vee \rangle < 2p^r(h-1)\}.$$

Notice that the category \mathcal{C} differs slightly from Jantzen's p^r -bounded category where it is assumed that $\langle \lambda, \alpha_0^\vee \rangle < 2p^r(h-1)$ [Jan1, p. 360]. Let $\mathcal{F}_\mathcal{C}$ be the truncation functor from $\text{Mod}(G)$ to $\text{Mod}(\mathcal{C})$ which takes $M \in \text{Mod}(G)$ to the largest submodule of M in \mathcal{C} . Note that this functor takes finite-dimensional modules to finite-dimensional modules. Let $\mathcal{G} = \mathcal{F}_\mathcal{C} \circ \text{ind}_{G(\mathbb{F}_q)}^G$ and $R^j \mathcal{G}$ be the higher right derived functors of \mathcal{G} . For $M \in \mathcal{C}$ and $N \in \text{Mod}(G(\mathbb{F}_q))$ there exists a first quadrant spectral sequence [BNP1, Thm. 4.4a]

$$E_2^{i,j} = \text{Ext}_G^i(M, R^j \mathcal{G}(N)) \Rightarrow \text{Ext}_{G(\mathbb{F}_q)}^{i+j}(M, N). \quad (2.1.1)$$

By analyzing this spectral sequence the following result [BNP1, Cor. 5.3i] was obtained which relates extensions of simple modules for $G(\mathbb{F}_q)$ with extensions in $\text{Mod}(G)$.

$$(2.1.2) \text{ If } p \geq 2(h-1) \text{ and } \lambda, \mu \in X_r(T) \text{ then } \text{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) \cong \text{Ext}_G^1(L(\lambda), \mathcal{G}(L(\mu))).$$

Interestingly enough, one can improve this result for all primes by imposing a condition on the highest weights of the simple modules [BNP1, Thm. 5.5a].

$$(2.1.3) \text{ If } \lambda, \mu \in X_r(T) \text{ with } \mu \not\geq \lambda \text{ then } \text{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) \cong \text{Ext}_G^1(L(\lambda), \mathcal{G}(L(\mu))).$$

2.2 The result below is an improved version of statement (2.1.3) because the image $\mathcal{G}(k)$ is much easier to describe than $\mathcal{G}(L(\mu))$ for μ arbitrary. Indeed, for $p \geq 3(h-1)$, the module $\mathcal{G}(k)$ was shown in [BNP1, Thm. 7.4] to be semisimple.

Theorem . *Let $\lambda, \mu \in X_r(T)$. Then $\text{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) \cong \text{Ext}_G^1(L(\lambda), L(\mu) \otimes \mathcal{G}(k))$.*

Proof. Since $\text{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) \cong \text{Ext}_{G(\mathbb{F}_q)}^1(L(-w_0\mu), L(-w_0\lambda))$ we may assume, without loss of generality, that $\langle \mu, \alpha_0^\vee \rangle \leq \langle \lambda, \alpha_0^\vee \rangle$. Furthermore, we may assume that $\langle \mu, \alpha_0^\vee \rangle = \langle \lambda, \alpha_0^\vee \rangle$ implies $\mu \not\geq \lambda$. Clearly it follows now for $\mu \neq \lambda$ that $\mu \not\geq \lambda$.

According to [P1, Lem. 1.4] we have the following statement. Let $\lambda \in X_r(T)$ and let N be a $G(\mathbb{F}_q)$ -module that contains only simple composition factors whose p^r -restricted highest weights γ satisfy $\gamma \leq (p^r - 1)\rho + w_0\lambda$ where w_0 is the long element in the Weyl group. Let $\text{St}_r = L((p^r - 1)\rho)$ be the Steinberg module for G_r (see [Jan1, p. 225]). Then the $G(\mathbb{F}_q)$ -head of $\text{St}_r \otimes N$ contains only simple modules $L(\sigma)$ whose highest weights σ are p^r -restricted and satisfy $\sigma \geq \lambda$.

We set $M = \text{St}_r \otimes L((p^r - 1)\rho + w_0\lambda)$ and use the preceding fact to obtain:

$$\text{Hom}_{G(\mathbb{F}_q)}(M \otimes L(-w_0\mu), k) \cong \text{Hom}_{G(\mathbb{F}_q)}(M, L(\mu)) \cong \begin{cases} k & \text{if } \lambda = \mu \\ 0 & \text{else.} \end{cases}$$

Either case yields an isomorphism

$$\text{Hom}_{G(\mathbb{F}_q)}(L(\lambda) \otimes L(-w_0\mu), k) \cong \text{Hom}_{G(\mathbb{F}_q)}(M \otimes L(-w_0\mu), k). \quad (2.2.1)$$

From [Jan1, II 10.15] one has $\text{Hom}_G(M, L(\lambda)) \cong k$. We define the G -module R via the following short exact sequence of G -modules

$$0 \rightarrow R \rightarrow \text{St}_r \otimes L((p^r - 1)\rho + w_0\lambda) \otimes L(-w_0\mu) \rightarrow L(\lambda) \otimes L(-w_0\mu) \rightarrow 0. \quad (2.2.2)$$

Note that this is a short exact sequence in the category \mathcal{C} because $\langle \mu, \alpha_0^\vee \rangle \leq \langle \lambda, \alpha_0^\vee \rangle$ and $w_0\alpha_0 = -\alpha_0$ implies that $\text{St}_r \otimes L((p^r - 1)\rho + w_0\lambda) \otimes L(-w_0\mu)$ is in \mathcal{C} . By using the fact that the Steinberg module is projective over $G(\mathbb{F}_q)$ we obtain the following exact sequence:

$$\begin{aligned} 0 &\rightarrow \text{Hom}_{G(\mathbb{F}_q)}(L(\lambda) \otimes L(-w_0\mu), k) \rightarrow \text{Hom}_{G(\mathbb{F}_q)}(\text{St}_r \otimes L((p^r - 1)\rho + w_0\lambda) \otimes L(-w_0\mu), k) \\ &\rightarrow \text{Hom}_{G(\mathbb{F}_q)}(R, k) \rightarrow \text{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda) \otimes L(-w_0\mu), k) \rightarrow 0. \end{aligned}$$

It follows from this exact sequence and the isomorphism in (2.2.1) that

$$\mathrm{Hom}_{G(\mathbb{F}_q)}(R, k) \cong \mathrm{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda) \otimes L(-w_0\mu), k) \cong \mathrm{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu)). \quad (2.2.3)$$

We next use the isomorphism in (2.2.3) to show that there is an embedding

$$\mathrm{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) \hookrightarrow \mathrm{Ext}_G^1(L(\lambda), L(\mu) \otimes \mathcal{G}(k)).$$

First, it follows from (2.2.1) and adjointness that

$$\mathrm{Hom}_G(L(\lambda) \otimes L(-w_0\lambda), \mathcal{G}(k)) \cong \mathrm{Hom}_G(M \otimes L(-w_0\lambda), \mathcal{G}(k)). \quad (2.2.4)$$

From the short exact sequence (2.2.2), one also obtains the following long exact sequence:

$$\begin{aligned} 0 &\rightarrow \mathrm{Hom}_G(L(\lambda) \otimes L(-w_0\mu), \mathcal{G}(k)) \rightarrow \mathrm{Hom}_G(\mathrm{St}_r \otimes L((p^r - 1)\rho + w_0\lambda) \otimes L(-w_0\mu), \mathcal{G}(k)) \\ &\rightarrow \mathrm{Hom}_G(R, \mathcal{G}(k)) \rightarrow \mathrm{Ext}_G^1(L(\lambda) \otimes L(-w_0\mu), \mathcal{G}(k)) \\ &\rightarrow \mathrm{Ext}_G^1(\mathrm{St}_r \otimes L((p^r - 1)\rho + w_0\lambda) \otimes L(-w_0\mu), \mathcal{G}(k)) \rightarrow \cdots \end{aligned}$$

From this sequence and (2.2.4), we get an injective map

$$\mathrm{Hom}_G(R, \mathcal{G}(k)) \hookrightarrow \mathrm{Ext}_G^1(L(\lambda) \otimes L(-w_0\mu), \mathcal{G}(k)).$$

But, $R \in \mathcal{C}$ so by adjointness we have

$$\mathrm{Hom}_{G(\mathbb{F}_q)}(R, k) \cong \mathrm{Hom}_G(R, \mathcal{G}(k)) \hookrightarrow \mathrm{Ext}_G^1(L(\lambda) \otimes L(-w_0\mu), \mathcal{G}(k)).$$

It follows from the isomorphism in (2.2.3) that there exists an injective map

$$\mathrm{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda) \otimes L(-w_0\mu), k) \hookrightarrow \mathrm{Ext}_G^1(L(\lambda) \otimes L(-w_0\mu), \mathcal{G}(k)).$$

On the other hand, the five term exact sequence of the spectral sequence in (2.1.1) [Jan1, I 4.1] shows that

$$\dim \mathrm{Ext}_G^1(L(\lambda) \otimes L(-w_0\mu), \mathcal{G}(k)) = \dim E_2^{1,0} \leq \dim E^1 = \dim \mathrm{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda) \otimes L(-w_0\mu), k)$$

so the preceding map is an isomorphism. Therefore, one obtains the desired isomorphism

$$\begin{aligned} \mathrm{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) &\cong \mathrm{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda) \otimes L(-w_0\mu), k) \\ &\cong \mathrm{Ext}_G^1(L(\lambda) \otimes L(-w_0\mu), \mathcal{G}(k)) \\ &\cong \mathrm{Ext}_G^1(L(\lambda), L(\mu) \otimes \mathcal{G}(k)). \end{aligned}$$

□

In Theorem 2.5, we will use the decomposition of $\mathcal{G}(k)$ from [BNP1] to further identify $\mathrm{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu))$. We first present some technical results which will simplify this identification.

2.3 For rational G -modules M and N , $\mathrm{Hom}_{G_r}(M, N)$ has a G -module structure. The following lemma gives some conditions under which the G -structure on $\mathrm{Hom}_{G_r}(L(\lambda), L(\mu) \otimes L(\nu))$ is in fact trivial. This fact will be used in later arguments on extensions.

Lemma . *Let $p \geq 2(h - 1)$ and M be a finite-dimensional rational G -module in Jantzen's p^r -bounded category [Jan1, p. 360].*

- (a) If the G -socle of M contains only p^r -restricted highest weights, then $\text{soc}_G M = \text{soc}_{G_r} M$.
- (b) If $\lambda, \mu, \nu \in X_r(T)$ with $\langle \nu, \alpha_0^\vee \rangle < p^r$ then

$$\text{Hom}_{G_r}(L(\lambda), L(\mu) \otimes L(\nu)) \cong \text{Hom}_G(L(\lambda), L(\mu) \otimes L(\nu)).$$

Proof. (a) By assumption, $\text{soc}_G M \cong \oplus_i L(\sigma_i)$ for some σ_i which are p^r -restricted. Since $p \geq 2(h-1)$, by [Jan1, II 11.11], the injective hull (and projective cover) $Q_r(\sigma_i)$ of each $L(\sigma_i)$ over G_r admits a unique G -module structure. Further, each $Q_r(\sigma_i)$ is injective in the p^r -bounded category (cf. [Jan1, II 11.11]). Hence, there is an embedding of G -modules $M \hookrightarrow \oplus_i Q_r(\sigma_i)$ which is an isomorphism on G -socles. Recall that $\text{soc}_G Q_r(\sigma_i) = L(\sigma_i) = \text{soc}_{G_r}(Q_r(\sigma_i))$ and hence one obtains the assertion via

$$\text{soc}_{G_r} M \subseteq \oplus_i \text{soc}_{G_r} Q_r(\sigma_i) = \oplus_i \text{soc}_G Q_r(\sigma_i) \cong \text{soc}_G M \subseteq \text{soc}_{G_r} M.$$

(b) It was shown in [BK, Theorem B] that the G -socle of $L(\mu) \otimes L(\nu)$ contains only simple modules with p^r -restricted highest weights. The assertion follows immediately from part (a) because $L(\mu) \otimes L(\nu)$ is p^r -bounded. \square

2.4 The following proposition is a generalization of Lemma 5.1 and Proposition 5.5 in [And1]. It will be used to simplify the formula in Theorem 2.5.

Proposition . *Assume $p \geq 2(h-1)$, $\lambda, \mu \in X_r(T)$, and $\nu_1, \nu_2 \in X(T)_+$ satisfy $\langle \nu_i, \alpha_0^\vee \rangle < 2(h-1)$, for $i = 1, 2$. Then the following hold:*

- (a) $\text{Ext}_G^1(L(\lambda) \otimes L(\nu_1)^{(r)}, L(\mu) \otimes L(\nu_2)) \cong \text{Hom}_{G/G_r}(L(\nu_1)^{(r)}, \text{Ext}_{G_r}^1(L(\lambda), L(\mu) \otimes L(\nu_2)))$.
- (b) If $p^r \xi$ is a weight of $\text{Ext}_{G_r}^1(L(\lambda), L(\mu) \otimes L(\nu_2))$ then $\langle \xi, \alpha_0^\vee \rangle \leq h-1$. In particular, $\text{Ext}_{G_r}^1(L(\lambda), L(\mu) \otimes L(\nu_2))$ is semisimple.
- (c) If $\text{Ext}_G^1(L(\lambda) \otimes L(\nu_1)^{(r)}, L(\mu) \otimes L(\nu_2)) \neq 0$, then $\langle \nu_1, \alpha_0^\vee \rangle \leq h-1$.

Proof. (a) The Lyndon-Hochschild-Serre spectral sequence for $G_r \trianglelefteq G$ yields

$$E_2^{i,j} = \text{Ext}_{G/G_r}^i(k, \text{Ext}_{G_r}^j(L(\lambda) \otimes L(\nu_1)^{(r)}, L(\mu) \otimes L(\nu_2))) \Rightarrow \text{Ext}_G^{i+j}(L(\lambda) \otimes L(\nu_1)^{(r)}, L(\mu) \otimes L(\nu_2)).$$

From Lemma 2.3b and the fact that $\text{Hom}_{G_r}(L(\lambda), L(\mu) \otimes L(\nu_2))$ is a trivial G -module, we have an isomorphism

$$\text{Ext}_{G/G_r}^i(L(\nu_1)^{(r)}, \text{Hom}_{G_r}(L(\lambda), L(\mu) \otimes L(\nu_2))) \cong \text{Ext}_G^i(L(\nu_1), k) \otimes \text{Hom}_G(L(\lambda), L(\mu) \otimes L(\nu_2)).$$

Any weight ν_1 that is linked to the zero weight has to satisfy $\langle \nu_1, \alpha_0^\vee \rangle \geq 2(p-h+1)$ [Jan2, 4.1]. The size of p forces $\text{Ext}_G^i(L(\nu_1), k) = 0$ for $i > 0$. Hence, from the five term exact sequence and the fact that $E_2^{1,0} = E_2^{2,0} = 0$, there is an isomorphism

$$\text{Ext}_G^1(L(\lambda) \otimes L(\nu_1)^{(r)}, L(\mu) \otimes L(\nu_2)) = E^1 \cong E_2^{0,1} = \text{Hom}_{G/G_r}(L(\nu_1)^{(r)}, \text{Ext}_{G_r}^1(L(\lambda), L(\mu) \otimes L(\nu_2))).$$

(b) By [Jan1, II 10.15] there is an embedding of G -modules $L(\mu) \hookrightarrow St_r \otimes L((p^r-1)\rho + w_0\mu)$. Therefore, we can define a G -module M via the exact sequence

$$0 \rightarrow L(\mu) \otimes L(\nu_2) \rightarrow St_r \otimes L((p^r-1)\rho + w_0\mu) \otimes L(\nu_2) \rightarrow M \rightarrow 0.$$

Consider the associated long exact sequence

$$\cdots \rightarrow \text{Hom}_{G_r}(L(\lambda), St_r \otimes L((p^r-1)\rho + w_0\mu) \otimes L(\nu_2)) \rightarrow \text{Hom}_{G_r}(L(\lambda), M)$$

$\rightarrow \text{Ext}_{G_r}^1(L(\lambda), L(\mu) \otimes L(\nu_2)) \rightarrow \text{Ext}_{G_r}^1(L(\lambda), St_r \otimes L((p^r - 1)\rho + w_0\mu) \otimes L(\nu_2)) \rightarrow \dots$

Using the injectivity of $St_r \otimes L((p^r - 1)\rho + w_0\mu) \otimes L(\nu_2)$ we obtain a surjection

$$\text{Hom}_{G_r}(L(\lambda), M) \rightarrow \text{Ext}_{G_r}^1(L(\lambda), L(\mu) \otimes L(\nu_2)).$$

Now assume that $p^r\xi$ is a weight of the G -module $\text{Ext}_{G_r}^1(L(\lambda), L(\mu) \otimes L(\nu_2))$. Then it is also a weight of the G -module $\text{Hom}_{G_r}(L(\lambda), M)$ and $\lambda + p^r\xi$ must be a weight of M . The highest weight of each $St_r \otimes L((p^r - 1)\rho + w_0\mu) \otimes L(\nu_2)$ is $2(p^r - 1)\rho + w_0\mu + \nu_2$. We obtain $\lambda + p^r\xi \leq 2(p^r - 1)\rho + w_0\mu + \nu_2$ or

$$p^r\xi \leq 2(p^r - 1)\rho - \lambda + w_0\mu + \nu_2. \quad (2.4.1)$$

On the other hand, $L(\mu) \otimes L(\nu_2)$ has a filtration of simple modules with not necessarily p^r -restricted highest weights $\gamma = \gamma_0 + p^r\gamma_1$ with $\gamma \leq \mu + \nu_2$. From [And2, Lem. 2.3] we obtain the following: for any γ and any highest weight $p^r\xi$ of $\text{Ext}_{G_r}^1(L(\lambda), L(\gamma_0) \otimes L(\gamma_1)^{(r)}) \cong \text{Ext}_{G_r}^1(L(\lambda), L(\gamma_0)) \otimes L(\gamma_1)^{(r)}$ there exists a simple root α with $p^r\xi \leq -w_0\lambda + \gamma_0 + p^r\gamma_1 + p^{r-1}\alpha \leq -w_0\lambda + \mu + \nu_2 + p^{r-1}\alpha$. For any weight $p^r\xi$ of $\text{Ext}_{G_r}^1(L(\lambda), L(\mu) \otimes L(\nu_2))$ we obtain the inequality

$$p^r\xi \leq -w_0\lambda + \mu + \nu_2 + p^{r-1}\alpha. \quad (2.4.2)$$

Adding equations (2.4.1) and (2.4.2) yields

$$2p^r\xi \leq 2(p^r - 1)\rho - \lambda - w_0\lambda + \mu + w_0\mu + 2\nu_2 + p^{r-1}\alpha.$$

Taking the inner product with α_0 and dividing both sides by 2 yields $p^r\langle \xi, \alpha_0^\vee \rangle < p^r(h - 1) + (h - 1) + p^{r-1}$. As long as $p \geq h$ we have $\langle \xi, \alpha_0^\vee \rangle \leq (h - 1)$. Since $p \geq 2(h - 1)$ the highest weights of all composition factors of $\text{Ext}_{G_r}^1(L(\lambda), L(\mu) \otimes L(\nu_2))^{(-1)}$ lie inside the closure of the lowest alcove. Hence, it is semisimple by the Linkage Principle.

(c) This follows immediately from parts (a) and (b). \square

2.5 In [BNP1, Thm. 7.4] it was shown that $\mathcal{G}(k)$ is semisimple for $p \geq 3(h - 1)$. Moreover, an explicit description of the module was given. In general the structure of $\mathcal{G}(k)$ can be quite complicated and is not necessarily semisimple. Using Theorem 2.2 and the description of $\mathcal{G}(k)$ for $p \geq 3(h - 1)$, we obtain the following Ext^1 -formula between simple $G(\mathbb{F}_q)$ -modules.

Theorem . For $p \geq 3(h - 1)$ and $\lambda, \mu \in X_r(T)$,

$$\text{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) \cong \bigoplus_{\nu \in \Gamma} \text{Ext}_G^1(L(\lambda) \otimes L(\nu)^{(r)}, L(\mu) \otimes L(\nu)),$$

where $\Gamma = \{\nu \in X(T)_+ \mid \langle \nu, \alpha_0^\vee \rangle < h\}$.

Proof. According to [BNP1, Thm. 7.4], if $p \geq 3(h - 1)$, the module $\mathcal{G}(k)$ is semisimple and

$$\mathcal{G}(k) = \bigoplus_{\nu \in \Gamma_{2h-1}} L(\nu - w_0p^r\nu) \cong \bigoplus_{\nu \in \Gamma_{2h-1}} L(\nu) \otimes L(-w_0\nu)^{(r)},$$

where $\Gamma_{2h-1} = \{\nu \in X(T)_+ \mid \langle \nu, \alpha_0^\vee \rangle < 2h - 1\}$. Hence, from Theorem 2.2, $\text{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu))$ may be identified in terms of a direct sum of G -extensions of the form

$$\text{Ext}_G^1(L(\lambda), L(\mu) \otimes L(\nu) \otimes L(-w_0\nu)^{(r)}) \cong \text{Ext}_G^1(L(\lambda) \otimes L(\nu)^{(r)}, L(\mu) \otimes L(\nu)).$$

Therefore, by Proposition 2.4c,

$$\begin{aligned} \mathrm{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) &\cong \bigoplus_{\nu \in \Gamma_{2h-1}} \mathrm{Ext}_G^1(L(\lambda) \otimes L(\nu)^{(r)}, L(\mu) \otimes L(\nu)) \\ &\cong \bigoplus_{\nu \in \Gamma} \mathrm{Ext}_G^1(L(\lambda) \otimes L(\nu)^{(r)}, L(\mu) \otimes L(\nu)). \end{aligned}$$

□

If either λ or μ are $(h-1)$ -deep inside an alcove, i.e. have distance at least $h-1$ from any alcove wall, the formula in part (a) of the Theorem immediately implies Theorem 3.2 in [And2] with significantly weaker deepness conditions (see also [BNP1, Thm. 7.6]). It explains the observations by Andersen and Ye in [Y] that Andersen's formula seemed to work for weights much closer to the alcove wall than predicted. It is also interesting to note that our Ext^1 -formula has similarities to Jantzen's [Jan3] and Chastkofsky's [Cha] formula for decomposing projective indecomposable G_r -modules as modules over $G(\mathbb{F}_q)$:

$$[Q_r(\lambda) : U_r(\mu)] = \sum_{\nu \in \Gamma} [L(\mu) \otimes L(\nu) : L(\lambda) \otimes L(\nu)^{(r)}]_G.$$

Here $U_r(\mu)$ denotes the projective cover of $L(\mu)$ as a $kG(\mathbb{F}_q)$ -module.

3 Extensions for $r \geq 2$

3.1 In this section, we strengthen Theorem 2.5 in the case that $r \geq 2$ (see Theorem 3.2) and use this to show the vanishing of self-extensions in Section 3.4. Throughout this section we use the following notation. Any $\sigma \in X_r(T)$ may be expressed as $\sigma = \sigma_0 + \sigma_1 p + \cdots + \sigma_{r-1} p^{r-1}$ where $\sigma_i \in X_1(T)$ for $0 \leq i \leq r-1$. For $r \geq 2$, set $\widehat{\sigma} = \sigma_0 + \sigma_1 p + \cdots + \sigma_{r-2} p^{r-2}$ so that $\sigma = \widehat{\sigma} + \sigma_{r-1} p^{r-1}$. The following technical result will be used in the proof of Theorem 3.2.

Proposition . *Let $p \geq 2(h-1)$, $\lambda, \mu \in X_r(T)$, and $\nu_1, \nu_2 \in X(T)_+$. If $r \geq 2$, $\nu_1 \neq 0$, and $\nu_2 \in \Gamma$, then*

$$\mathrm{Hom}_{G/G_{r-1}}(L(\lambda_{r-1})^{(r-1)} \otimes L(\nu_1)^{(r)}, \mathrm{Ext}_{G_{r-1}}^1(L(\widehat{\lambda}), L(\widehat{\mu}) \otimes L(\nu_2)) \otimes L(\mu_{r-1})^{(r-1)}) = 0.$$

Proof. Using duality as discussed in the last paragraph of Section 1.2, we obtain

$$\begin{aligned} \mathrm{Hom}_{G/G_{r-1}}(L(\lambda_{r-1})^{(r-1)} \otimes L(\nu_1)^{(r)}, \mathrm{Ext}_{G_{r-1}}^1(L(\widehat{\lambda}), L(\widehat{\mu}) \otimes L(\nu_2)) \otimes L(\mu_{r-1})^{(r-1)}) &\cong \\ \mathrm{Hom}_{G/G_{r-1}}(L(-w_0 \mu_{r-1})^{(r-1)} \otimes L(\nu_1)^{(r)}, \mathrm{Ext}_{G_{r-1}}^1(L(\widehat{\lambda}), L(\widehat{\mu}) \otimes L(\nu_2)) \otimes L(-w_0 \lambda_{r-1})^{(r-1)}), & \\ \text{and hence may assume that } \langle \mu_{r-1}, \alpha_0^\vee \rangle \leq \langle \lambda_{r-1}, \alpha_0^\vee \rangle. & \end{aligned}$$

By Proposition 2.4b, any highest weight ξ of $\mathrm{Ext}_{G_{r-1}}^1(L(\widehat{\lambda}), L(\widehat{\mu}) \otimes L(\nu_2))$ satisfies $\langle \xi, \alpha_0^\vee \rangle \leq p^{r-1}(h-1)$. Any highest weight η of $\mathrm{Ext}_{G_{r-1}}^1(L(\widehat{\lambda}), L(\widehat{\mu}) \otimes L(\nu_2)) \otimes L(\mu_{r-1})^{(r-1)}$ therefore satisfies $\langle \eta, \alpha_0^\vee \rangle \leq p^{r-1}(\langle \mu_{r-1}, \alpha_0^\vee \rangle + h-1)$. By Steinberg's Tensor Product Theorem, the G -module $L(\lambda_{r-1})^{(r-1)} \otimes L(\nu_1)^{(r)}$ may be identified with the simple module $L(p^{r-1} \lambda_{r-1} + p^r \nu_1)$. Hence, the desired Hom-group is clearly zero unless

$$\langle p^{r-1} \lambda_{r-1} + p^r \nu_1, \alpha_0^\vee \rangle \leq p^{r-1}(\langle \mu_{r-1}, \alpha_0^\vee \rangle + h-1).$$

Using the assumptions $\langle \mu_{r-1}, \alpha_0^\vee \rangle \leq \langle \lambda_{r-1}, \alpha_0^\vee \rangle$ and $\nu_1 \neq 0$, one obtains $p^r \leq p^r \langle \nu_1, \alpha_0^\vee \rangle \leq p^{r-1}(h-1)$. Hence, with $p \geq 2(h-1)$, one obtains a contradiction. □

3.2 The theorem below shows that Theorem 2.5 can be significantly strengthened for $r \geq 2$. In particular, extensions of simple $G(\mathbb{F}_q)$ modules are determined by G -extensions of the same simple modules, the socle of extensions of simple G_1 -modules, and decomposition of tensor products of simple G -modules.

Theorem . For $p \geq 3(h-1)$, $r \geq 2$, and $\lambda, \mu \in X_r(T)$,

$$\mathrm{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) \cong \mathrm{Ext}_G^1(L(\lambda), L(\mu)) \oplus R \quad (3.2.1)$$

where

$$R = \bigoplus_{\nu \in \Gamma - \{0\}} \mathrm{Hom}_G(L(\nu), \mathrm{Ext}_{G_1}^1(L(\lambda_{r-1}), L(\mu_{r-1}))^{(-1)}) \otimes \mathrm{Hom}_G(L(\widehat{\lambda}), L(\widehat{\mu}) \otimes L(\nu)). \quad (3.2.2)$$

Proof. Without loss of generality we may assume that $\langle \mu, \alpha_0^\vee \rangle \leq \langle \lambda, \alpha_0^\vee \rangle$ because

$$\mathrm{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) \cong \mathrm{Ext}_{G(\mathbb{F}_q)}^1(L(-w_0\mu), L(-w_0\lambda)).$$

From Theorem 2.5, the remainder term R may be identified as

$$R = \bigoplus_{\nu \in \Gamma - \{0\}} \mathrm{Ext}_G^1(L(\lambda) \otimes L(\nu)^{(r)}, L(\mu) \otimes L(\nu)).$$

We proceed to identify $\mathrm{Ext}_G^1(L(\lambda) \otimes L(\nu)^{(r)}, L(\mu) \otimes L(\nu))$ for ν in $\Gamma - \{0\}$. We begin by applying the Lyndon-Hochschild-Serre spectral sequence for $G_{r-1} \trianglelefteq G$:

$$\begin{aligned} E_2^{i,j} &= \mathrm{Ext}_{G/G_{r-1}}^i(L(\lambda_{r-1})^{(r-1)} \otimes L(\nu)^{(r)}, \mathrm{Ext}_{G_{r-1}}^j(L(\widehat{\lambda}), L(\widehat{\mu}) \otimes L(\nu)) \otimes L(\mu_{r-1})^{(r-1)}) \\ &\Rightarrow \mathrm{Ext}_G^{i+j}(L(\lambda) \otimes L(\nu)^{(r)}, L(\mu) \otimes L(\nu)). \end{aligned}$$

We have $E^1 = \mathrm{Ext}_G^1(L(\lambda) \otimes L(\nu)^{(r)}, L(\mu) \otimes L(\nu))$. By Proposition 3.1, it follows that $E_2^{0,1} = 0$ and hence $E^1 \cong E_2^{1,0}$ (from the beginning of the corresponding five term exact sequence $0 \rightarrow E_2^{1,0} \rightarrow E^1 \rightarrow E_2^{0,1} \rightarrow E_2^{2,0} \rightarrow E^2$). Therefore, with Lemma 2.3b, we have the following isomorphisms:

$$\begin{aligned} E^1 &\cong \mathrm{Ext}_{G/G_{r-1}}^1(L(\lambda_{r-1})^{(r-1)} \otimes L(\nu)^{(r)}, \mathrm{Hom}_{G_{r-1}}(L(\widehat{\lambda}), L(\widehat{\mu}) \otimes L(\nu)) \otimes L(\mu_{r-1})^{(r-1)}) \\ &\cong \mathrm{Ext}_G^1(L(\lambda_{r-1}) \otimes L(\nu)^{(1)}, L(\mu_{r-1})) \otimes \mathrm{Hom}_G(L(\widehat{\lambda}), L(\widehat{\mu}) \otimes L(\nu)). \end{aligned}$$

Applying Proposition 2.4a to $\mathrm{Ext}_G^1(L(\lambda_{r-1}) \otimes L(\nu)^{(1)}, L(\mu_{r-1}))$, for any $\nu \in \Gamma - \{0\}$, we then have

$$\begin{aligned} E^1 &\cong \mathrm{Ext}_G^1(L(\lambda_{r-1}) \otimes L(\nu)^{(1)}, L(\mu_{r-1})) \otimes \mathrm{Hom}_G(L(\widehat{\lambda}), L(\widehat{\mu}) \otimes L(\nu)) \\ &\cong \mathrm{Hom}_{G/G_1}(L(\nu)^{(1)}, \mathrm{Ext}_{G_1}^1(L(\lambda_{r-1}), L(\mu_{r-1}))) \otimes \mathrm{Hom}_G(L(\widehat{\lambda}), L(\widehat{\mu}) \otimes L(\nu)) \\ &\cong \mathrm{Hom}_G(L(\nu), \mathrm{Ext}_{G_1}^1(L(\lambda_{r-1}), L(\mu_{r-1}))^{(-1)}) \otimes \mathrm{Hom}_G(L(\widehat{\lambda}), L(\widehat{\mu}) \otimes L(\nu)). \end{aligned}$$

The theorem now follows from Theorem 2.5. \square

3.3 It was shown in [CPSK, Thm. 7.4] and in [And2, Prop 2.7] that the restriction map in cohomology $\mathrm{res} : \mathrm{Ext}_G^1(L(\lambda), L(\mu)) \rightarrow \mathrm{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu))$ is an injective map. In [And2, Rem. 2.10(iii)] Andersen observed that for fixed λ and μ and for sufficiently large r this map is an isomorphism. Our formula in Theorem 3.2 captures this fact. For r large, and

λ, μ fixed, $\lambda_{r-1} = \mu_{r-1} = 0$, thus $R = 0$. Indeed, from Theorem 3.2, we deduce the following sufficient conditions for extensions between simple $G(\mathbb{F}_q)$ -modules to vanish.

Corollary . *Let $p \geq 3(h-1)$, $r \geq 2$, and $\lambda, \mu \in X_r(T)$. If*

- (a) $\lambda_{r-1} = \mu_{r-1}$ or
- (b) *there exists a root α such that $|\langle \widehat{\lambda} - \widehat{\mu}, \alpha^\vee \rangle| > h-1$,*

then $\text{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) \cong \text{Ext}_G^1(L(\lambda), L(\mu))$.

Proof. If $\lambda_{r-1} = \mu_{r-1}$, then $\text{Ext}_{G_1}^1(L(\lambda_{r-1}), L(\mu_{r-1})) = 0$ for $p > 2$ by [And1, Thm. 4.5]. Hence the remainder term R in Theorem 3.2 vanishes giving the claimed isomorphism.

In the second case, we embed $\text{Hom}_G(L(\widehat{\lambda}), L(\widehat{\mu}) \otimes L(\nu))$ in $\text{Hom}_G(V(\widehat{\lambda}), H^0(\widehat{\mu}) \otimes H^0(\nu))$. The module $H^0(\widehat{\mu}) \otimes H^0(\nu)$ has a good filtration with factors $H(\widehat{\mu} + \gamma)$ [Mat]. Moreover, $|\langle \gamma, \alpha^\vee \rangle| \leq \langle \nu, \alpha_0^\vee \rangle$ for any root α and $\langle \nu, \alpha_0^\vee \rangle < h$ because $\nu \in \Gamma$. The assumption on $\widehat{\lambda}$ and $\widehat{\mu}$ implies that $H^0(\widehat{\lambda})$ does not appear as a factor and by [Jan1, II 4.16] one obtains $\dim \text{Hom}_G(L(\widehat{\lambda}), L(\widehat{\mu}) \otimes L(\nu)) \leq \dim \text{Hom}_G(V(\widehat{\lambda}), H^0(\widehat{\mu}) \otimes H^0(\nu)) = 0$. Hence, $R = 0$ and the result follows. \square

3.4 Self-extensions A particular consequence of Corollary 3.3 is that self-extensions do not exist for simple $G(\mathbb{F}_q)$ -modules when $r \geq 2$ and $p \geq 3(h-1)$.

Theorem . *For $p \geq 3(h-1)$, $r \geq 2$, and $\lambda \in X_r(T)$, $\text{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\lambda)) = 0$.*

Proof. Since it is well known that $\text{Ext}_G^1(L(\lambda), L(\lambda)) = 0$ (cf. [Jan1, II 2.12]), this follows immediately from Corollary 3.3a. \square

4 Extensions for $r = 1$

4.1 In this section, we consider extensions over $G(\mathbb{F}_p)$. We begin with a modification of Theorem 2.5. As in the case when $r \geq 2$ (Theorem 3.2), this formula reduces the computation of these extension groups to computing extensions for the algebraic group G and for the first Frobenius kernel G_1 . This formula will not be used in the remainder of the paper.

Theorem . *Let $p \geq 3(h-1)$ and $\lambda, \mu \in X_1(T)$. Then*

$$\text{Ext}_{G(\mathbb{F}_p)}^1(L(\lambda), L(\mu)) \cong \text{Ext}_G^1(L(\lambda), L(\mu)) \oplus R \quad (4.1.1)$$

where

$$R = \bigoplus_{\nu \in \Gamma - \{0\}} \text{Hom}_G(L(\nu), \text{Ext}_{G_1}^1(L(\lambda), L(\mu) \otimes L(\nu))^{(-1)}). \quad (4.1.2)$$

Proof. By Theorem 2.5, the remainder term R may be identified as

$$R = \bigoplus_{\nu \in \Gamma - \{0\}} \text{Ext}_G^1(L(\lambda) \otimes L(\nu)^{(1)}, L(\mu) \otimes L(\nu)).$$

By Proposition 2.4a, we have

$$\text{Ext}_G^1(L(\lambda) \otimes L(\nu)^{(1)}, L(\lambda) \otimes L(\nu)) \cong \text{Hom}_G(L(\nu), \text{Ext}_{G_1}^1(L(\lambda), L(\mu) \otimes L(\nu))^{(-1)}).$$

□

4.2 The remainder of this section is devoted to analyzing self-extensions in the case $r = 1$. The analysis is considerably more subtle since self-extensions are known to exist for groups of type A_1 and C_n [Hum], [TZ, Remark 3.18], [P3]. The goal is to prove the following theorem, which states that self-extensions of simple modules can exist only in type A_1 and C_n for specific weights.

Theorem . *Let $p \geq 3(h - 1)$ and $\lambda \in X_1(T)$. If either*

- (a) *G does not have underlying root system of type A_1 or C_n or*
- (b) *$\langle \lambda, \alpha_n^\vee \rangle \neq \frac{p-2-c}{2}$, where α_n is the unique long simple root and c is odd with $0 < |c| \leq h - 1$,*

then $\text{Ext}_{G(\mathbb{F}_p)}^1(L(\lambda), L(\lambda)) = 0$.

The analysis requires a sequence of technical results. The strategy employed here is similar to that used by Andersen [And1] to study self-extensions over G_r . Note that we refer to Jantzen's presentation of this work in [Jan1] rather than the original source.

To begin, according to Theorem 2.5, for $p \geq 3(h - 1)$ and $\lambda \in X_1(T)$,

$$\text{Ext}_{G(\mathbb{F}_p)}^1(L(\lambda), L(\lambda)) \cong \bigoplus_{\nu \in \Gamma} \text{Ext}_G^1(L(\lambda) \otimes L(\nu)^{(1)}, L(\lambda) \otimes L(\nu)).$$

Suppose $\text{Ext}_{G(\mathbb{F}_p)}^1(L(\lambda), L(\lambda)) \neq 0$. Since $\text{Ext}_G^1(L(\lambda), L(\lambda)) = 0$ (cf. [Jan1, II 2.12]), there exists $\nu \in \Gamma - \{0\}$ such that $\text{Ext}_G^1(L(\lambda) \otimes L(\nu)^{(1)}, L(\lambda) \otimes L(\nu)) \neq 0$.

The next step is to reduce the problem to one over the Borel subgroup $B \subset G$ and its first Frobenius kernel B_1 . Indeed, it follows from Lemma 4.5 that if $\text{Ext}_G^1(L(\lambda) \otimes L(\nu)^{(1)}, L(\lambda) \otimes L(\nu)) \neq 0$ for some ν , then there exists a weight σ of $L(\nu)$ such that $\text{Hom}_{B/B_1}(p\sigma, \text{Ext}_{B_1}^1(V(\lambda), \lambda \otimes L(\nu))) \neq 0$. The results in Sections 4.3 and 4.4 are technical results used to obtain Lemma 4.5.

The final major step is to determine when the above Hom-group can be non-zero. That reduces to the question of when $\text{Ext}_{B_1}^1(V(\lambda), \lambda \otimes L(\nu))$ is non-zero. In Theorem 4.9, it is shown that this is non-zero only for specific weights λ when the underlying root system of G is of type A_1 or C_n . Sections 4.6-4.8 contain more technical results needed for the proof of Theorem 4.9. Our main theorem then follows from Corollary 4.9.

The reader will note that the results presented below are somewhat more general than needed to obtain the above self-extension result. In particular, rather than a single weight ν as above, possibly distinct weights ν_1 and ν_2 are used throughout this section. One source of motivation for this is the possible application of these results to twisted Chevalley groups which the authors plan to investigate in the future.

4.3 The following embeddings involving induced modules $H^0(\lambda) = \text{ind}_B^G \lambda$ and Weyl modules $V(\lambda) \cong H^0(-w_0\lambda)^*$ will be used in Lemma 4.5.

Lemma . *Let $\lambda \in X_1(T)$, $\nu_1, \nu_2 \in X(T)_+$ with $\nu_1 \neq 0$ and $\langle \nu_2, \alpha_0^\vee \rangle < p$. For $M \in \{L(\lambda), V(\lambda)\}$, there exist the following embeddings*

$$(a) \text{Ext}_G^1(M \otimes V(\nu_1)^{(1)}, L(\lambda) \otimes H^0(\nu_2)) \hookrightarrow \text{Ext}_G^1(M \otimes V(\nu_1)^{(1)}, H^0(\lambda) \otimes H^0(\nu_2));$$

$$(b) \operatorname{Ext}_G^1(L(\lambda) \otimes V(\nu_1)^{(1)}, L(\lambda) \otimes H^0(\nu_2)) \hookrightarrow \operatorname{Ext}_G^1(V(\lambda) \otimes V(\nu_1)^{(1)}, H^0(\lambda) \otimes H^0(\nu_2)).$$

Proof. (a) Consider the short exact sequence

$$0 \rightarrow L(\lambda) \otimes H^0(\nu_2) \rightarrow H^0(\lambda) \otimes H^0(\nu_2) \rightarrow Q \rightarrow 0$$

and the corresponding long exact sequence

$$\begin{aligned} \cdots \rightarrow \operatorname{Hom}_G(M \otimes V(\nu_1)^{(1)}, Q) &\rightarrow \operatorname{Ext}_G^1(M \otimes V(\nu_1)^{(1)}, L(\lambda) \otimes H^0(\nu_2)) \\ &\rightarrow \operatorname{Ext}_G^1(M \otimes V(\nu_1)^{(1)}, H^0(\lambda) \otimes H^0(\nu_2)) \rightarrow \cdots \end{aligned}$$

It suffices to show that $\operatorname{Hom}_G(M \otimes V(\nu_1)^{(1)}, Q) = 0$. Since the G -head of $M \otimes V(\nu_1)^{(1)}$ is $L(\lambda) \otimes L(\nu_1)^{(1)}$ and therefore simple, $M \otimes V(\nu_1)^{(1)}$ is a quotient of $V(\lambda + p\nu_1)$. Hence, if there exists a non-zero homomorphism $M \otimes V(\nu_1)^{(1)} \rightarrow Q$, then there exists a non-zero homomorphism $V(\lambda + p\nu_1) \rightarrow Q$. So $\lambda + p\nu_1$ is a weight of Q . Hence $\lambda + p\nu_1 \leq \lambda + \nu_2$ which would imply that $0 \neq p\langle \nu_1, \alpha_0^\vee \rangle \leq \langle \nu_2, \alpha_0^\vee \rangle < p$, which is impossible. (b) This part follows by using the duality $H^0(\lambda)^* \cong V(-w_0\lambda)$ and applying part (a) twice. \square

4.4 Let B be a Borel subgroup of G and B_1 be the first Frobenius kernel of B . The following technical result is used in several places.

Lemma . *Let $\lambda \in X_1(T)$, $\nu \in X(T)_+$ with $\langle \nu, \alpha_0^\vee \rangle < p$, and $\gamma \in X(T)_+$ with $\lambda + \gamma$ dominant and $\langle \gamma, \alpha^\vee \rangle < p$ for any root α . Then the following hold:*

- (a) $\operatorname{Hom}_{B_1}(V(\lambda + \gamma), \lambda) \cong \begin{cases} k & \text{if } \gamma = 0 \\ 0 & \text{else.} \end{cases}$
- (b) *As a B -module, $\operatorname{Hom}_{B_1}(V(\lambda), \lambda \otimes H^0(\nu))$ is a direct sum of trivial modules.*

Proof. (a) Suppose that $\operatorname{Hom}_{B_1}(V(\lambda + \gamma), \lambda) \neq 0$. Then there exists $\sigma \in X(T)$ such that $\operatorname{Hom}_{B/B_1}(p\sigma, \operatorname{Hom}_{B_1}(V(\lambda + \gamma), \lambda)) \neq 0$. But,

$$\begin{aligned} \operatorname{Hom}_{B/B_1}(p\sigma, \operatorname{Hom}_{B_1}(V(\lambda + \gamma), \lambda)) &\cong \operatorname{Hom}_{B/B_1}(k, \operatorname{Hom}_{B_1}(V(\lambda + \gamma), \lambda - p\sigma)) \\ &\cong \operatorname{Hom}_B(V(\lambda + \gamma), \lambda - p\sigma). \end{aligned}$$

As a B -module, $V(\lambda + \gamma)$ has a unique one-dimensional quotient with weight $\lambda + \gamma$. Hence $\lambda + \gamma = \lambda - p\sigma$, so $\gamma = -p\sigma$. The size of γ forces $\sigma = 0$ and $\gamma = 0$.

(b) By duality $\operatorname{Hom}_{B_1}(V(\lambda), \lambda \otimes H^0(\nu)) \cong \operatorname{Hom}_{B_1}(V(\lambda) \otimes V(-w_0\nu), \lambda)$. By [Mat] $V(\lambda) \otimes V(-w_0\nu)$ has a Weyl filtration with factors $V(\lambda + \gamma)$. Any B -composition factor of $\operatorname{Hom}_{B_1}(V(\lambda) \otimes V(-w_0\nu), \lambda)$ has to appear in some $\operatorname{Hom}_{B_1}(V(\lambda + \gamma), \lambda)$. The condition on ν implies that γ satisfies the hypotheses. By part (a) the only composition factors of $\operatorname{Hom}_{B_1}(V(\lambda + \gamma), \lambda)$ are trivial modules. Since there are no self-extensions of the trivial module as a B -module, the assertion follows. \square

4.5 The following lemma will allow us to reduce the problem to one for B - and B_1 -cohomology.

Lemma . *Let $\lambda \in X_1(T)$ and $\nu_1, \nu_2 \in X(T)_+$ with $\nu_1 \neq 0$ and $\langle \nu_2, \alpha_0^\vee \rangle < p$. Suppose that $\text{Ext}_G^1(L(\lambda) \otimes V(\nu_1)^{(1)}, L(\lambda) \otimes H^0(\nu_2)) \neq 0$. Then there exists a weight σ of $V(\nu_1)$ such that*

$$\text{Hom}_{B/B_1}(p\sigma, \text{Ext}_{B_1}^1(V(\lambda), \lambda \otimes H^0(\nu_2))) \neq 0.$$

Proof. By Lemma 4.3b, it follows that

$$\text{Ext}_G^1(V(\lambda) \otimes V(\nu_1)^{(1)}, H^0(\lambda) \otimes H^0(\nu_2)) \neq 0.$$

By Frobenius reciprocity, we obtain

$$\text{Ext}_B^1(V(\lambda) \otimes V(\nu_1)^{(1)}, \lambda \otimes H^0(\nu_2)) \neq 0.$$

Consider the Lyndon-Hochschild-Serre spectral sequence for $B_1 \trianglelefteq B$:

$$E_2^{i,j} = \text{Ext}_{B/B_1}^i(V(\nu_1)^{(1)}, \text{Ext}_{B_1}^j(V(\lambda), \lambda \otimes H^0(\nu_2))) \Rightarrow \text{Ext}_B^{i+j}(V(\lambda) \otimes V(\nu_1)^{(1)}, \lambda \otimes H^0(\nu_2))$$

and the five-term exact sequence

$$\begin{aligned} 0 &\rightarrow \text{Ext}_{B/B_1}^1(V(\nu_1)^{(1)}, \text{Hom}_{B_1}(V(\lambda), \lambda \otimes H^0(\nu_2))) \rightarrow \text{Ext}_B^1(V(\lambda) \otimes V(\nu_1)^{(1)}, \lambda \otimes H^0(\nu_2)) \\ &\rightarrow \text{Hom}_{B/B_1}(V(\nu_1)^{(1)}, \text{Ext}_{B_1}^1(V(\lambda), \lambda \otimes H^0(\nu_2))) \rightarrow \text{Ext}_{B/B_1}^2(V(\nu_1), \text{Hom}_{B_1}(V(\lambda), \lambda \otimes H^0(\nu_2))). \end{aligned}$$

Lemma 4.4b shows that $\text{Hom}_{B_1}(V(\lambda), \lambda \otimes H^0(\nu_2))$ is a direct sum of trivial B -modules. So for $i \geq 1$ we have

$$\begin{aligned} E_2^{i,0} &= \text{Ext}_{B/B_1}^i(V(\nu_1)^{(1)}, \text{Hom}_{B_1}(V(\lambda), \lambda \otimes H^0(\nu_2))) \\ &= \bigoplus \text{Ext}_{B/B_1}^i(V(\nu_1)^{(1)}, k) \\ &\cong \bigoplus \text{Ext}_B^i(V(\nu_1), k) \\ &\cong \bigoplus \text{Ext}_G^i(V(\nu_1), k) \quad \text{by Frobenius reciprocity.} \end{aligned}$$

By [Jan1, II 4.13], we have $E_2^{i,0} = 0$ for $i \geq 1$. Consequently,

$$0 \neq E^1 \cong E_2^{0,1} \cong \text{Hom}_{B/B_1}(V(\nu_1)^{(1)}, \text{Ext}_{B_1}^1(V(\lambda), \lambda \otimes H^0(\nu_2))).$$

Therefore, there exists a weight σ of $V(\nu_1)$ such that $\text{Hom}_{B/B_1}(p\sigma, \text{Ext}_{B_1}^1(V(\lambda), \lambda \otimes H^0(\nu_2))) \neq 0$. \square

4.6 To study groups of the form $\text{Hom}_{B/B_1}(p\sigma, \text{Ext}_{B_1}^1(M, \lambda))$, we will repeatedly make use of the spectral sequence

$$E_2^{i,j} = \text{Ext}_{B/B_1}^i(p\sigma, \text{Ext}_{B_1}^j(M, \lambda)) \Rightarrow \text{Ext}_B^{i+j}(M, \lambda - p\sigma)$$

and its five-term exact sequence. The following result gives some conditions under which the abutment in such a spectral sequence can be non-zero. In what follows, we also make use of the Strong Linkage Principle [Jan1, II 6.13] and the refined order relation “ \uparrow ” on weights as defined in [Jan1, II 6.4].

Proposition . *Let $\lambda \in X_1(T)$ and $\gamma \in X(T)$ with $\lambda + \gamma$ dominant and $\langle \gamma, \beta^\vee \rangle < p/3$ for any root β . If there exists a weight σ with $\text{Ext}_B^1(V(\lambda + \gamma), \lambda - p\sigma) \neq 0$, then one of the following holds:*

- (a) $\gamma = 0$ and $\sigma = p^m \alpha$, where α denotes a simple root and m a non-negative integer,

- (b) σ is equal to a simple root α , $\langle \lambda + \rho, \alpha^\vee \rangle > p - p/6$, and the weight $\lambda + \gamma$ is the image of λ after reflection across the (α, p) -wall.
- (c) the underlying root system of G is of type A_1 or C_n , and
- (i) $\sigma = \frac{\alpha_n}{2}$, where α_n is the unique long (last) simple root,
 - (ii) $\langle \lambda, \alpha_n^\vee \rangle = \frac{p-2-c}{2}$, where c is an integer and $0 \leq |c| < p/3$,
 - (iii) $\gamma = \left(\frac{p}{2} - \langle \lambda + \rho, \alpha_n^\vee \rangle\right) \alpha_n$.

Proof. It follows from Frobenius reciprocity and [Jan1, II 4.13] that $\lambda - p\sigma \notin X(T)_+$. In particular, $\sigma \neq 0$. Consider the spectral sequence [Jan1, I 4.5]

$$E_2^{i,j} = \text{Ext}_G^i(V(\lambda + \gamma), R^j \text{ind}_B^G(\lambda - p\sigma)) \Rightarrow \text{Ext}_B^{i+j}(V(\lambda + \gamma), \lambda - p\sigma)$$

and the associated five term exact sequence

$$0 \rightarrow E_2^{1,0} \rightarrow E^1 \rightarrow E_2^{0,1} \rightarrow E_2^{2,0}.$$

Since $\lambda - p\sigma \notin X(T)_+$, $H^0(\lambda - p\sigma) = 0$, and so $E_2^{1,0} = 0 = E_2^{2,0}$. Hence we obtain the isomorphism

$$\text{Ext}_B^1(V(\lambda + \gamma), \lambda - p\sigma) = E^1 \cong E_2^{0,1} = \text{Hom}_G(V(\lambda + \gamma), H^1(\lambda - p\sigma)).$$

For this Hom-group to be non-zero, we must have $H^1(\lambda - p\sigma) \neq 0$, and so we apply [Jan1, II 5.4, 5.15] and obtain two possibilities.

Case 1: *There exists a dominant weight η and a simple root α such that $\lambda - p\sigma = s_\alpha \cdot \eta$ and $\langle \eta, \alpha^\vee \rangle < p - 1$. Moreover, $H^1(\lambda - p\sigma) \cong H^0(\eta)$.*

Since $\text{Hom}_G(V(\lambda + \gamma), H^0(\eta)) \neq 0$ it follows that $\eta = \lambda + \gamma$. Thus, the above equation becomes $\lambda - p\sigma = s_\alpha \cdot (\lambda + \gamma)$. Solving for $p\sigma$ yields

$$p\sigma = -\gamma + \langle \lambda + \gamma + \rho, \alpha^\vee \rangle \alpha. \quad (4.6.1)$$

Taking the inner product with α^\vee yields

$$p\langle \sigma, \alpha^\vee \rangle = -\langle \gamma, \alpha^\vee \rangle + 2\langle \lambda + \gamma + \rho, \alpha^\vee \rangle, \quad (4.6.2)$$

which can be written as

$$p\langle \sigma, \alpha^\vee \rangle = \langle \lambda + \rho, \alpha^\vee \rangle + \langle \lambda + \gamma + \rho, \alpha^\vee \rangle, \quad (4.6.3)$$

Now λ is restricted and $\langle \lambda + \gamma + \rho, \alpha^\vee \rangle = \langle \eta + \rho, \alpha^\vee \rangle < p$. That forces $\langle \sigma, \alpha^\vee \rangle = 1$. Next we assume that α is not the unique long simple root in a root system of type A_1 or C_n . Then there exists a positive root β with $\langle \alpha, \beta^\vee \rangle = -1$. We take the inner product of equation (4.6.1) with β^\vee and obtain

$$p\langle \sigma, \beta^\vee \rangle = -\langle \gamma, \beta^\vee \rangle - \langle \lambda + \gamma + \rho, \alpha^\vee \rangle. \quad (4.6.4)$$

Adding equation (4.6.4) twice to equation (4.6.2) yields

$$p + 2p\langle \sigma, \beta^\vee \rangle = -\langle \gamma, \alpha^\vee \rangle - 2\langle \gamma, \beta^\vee \rangle. \quad (4.6.5)$$

The absolute value of the right side of the equation is less than p while the left hand side is a non-zero multiple of p . We obtain a contradiction.

The root system is therefore of type A_1 or C_n and α is the unique long simple root which we denote by α_n . Taking the inner product of (4.6.1) with any of the simple roots

α_1 through α_{n-2} yields $p\langle\sigma, \alpha_i^\vee\rangle = -\langle\gamma, \alpha_i^\vee\rangle$ and thus $\langle\sigma, \alpha_i^\vee\rangle = 0$. The inner product of (4.6.1) with α_{n-1} yields

$$p\langle\sigma, \alpha_{n-1}^\vee\rangle = -\langle\gamma, \alpha_{n-1}^\vee\rangle - 2\langle\lambda + \gamma + \rho, \alpha_n^\vee\rangle. \quad (4.6.6)$$

Adding (4.6.6) to (4.6.2) yields

$$p\langle\sigma, \alpha_{n-1}^\vee\rangle + p = -\langle\gamma, \alpha_{n-1}^\vee\rangle - \langle\gamma, \alpha_n^\vee\rangle.$$

This forces $\langle\sigma, \alpha_{n-1}^\vee\rangle = -1$ and $\sigma = \alpha_n/2$. (Note that in type A_1 the latter conclusion follows immediately from $\langle\sigma, \alpha^\vee\rangle = 1$.) Now equation (4.6.1) can be re-written as

$$\gamma = -\frac{p}{2}\alpha_n + \langle\lambda + \rho, \alpha_n^\vee\rangle\alpha_n + \langle\gamma, \alpha_n^\vee\rangle\alpha_n \quad (4.6.7)$$

and (4.6.2) can be rewritten as

$$\langle\gamma, \alpha_n^\vee\rangle = p - 2\langle\lambda + \rho, \alpha_n^\vee\rangle. \quad (4.6.8)$$

Substituting equation (4.6.8) into equation (4.6.7), we get $\gamma = \left(\frac{p}{2} - \langle\lambda + \rho, \alpha_n^\vee\rangle\right)\alpha_n$. If p is odd and γ is non-zero, then $\lambda + \gamma$ is the reflection of λ across the hyperplane $\langle x + \rho, \alpha_n^\vee\rangle = p/2$. Finally, conclusion (c)(ii) follows from the description of γ by taking the inner product with α_n^\vee .

Case 2: *There exists a dominant weight η , a unique simple root α , and integers $n > 0$ and $0 < a < p$, such that*

$$\lambda - p\sigma = \eta - ap^n\alpha. \quad (4.6.9)$$

Moreover, it follows that $\text{soc}_G H^1(\lambda - p\sigma) \cong L(\eta)$, which implies by the Strong Linkage Principle that $\eta \uparrow \lambda + \gamma$.

We rewrite equation (4.6.9) in the form

$$\eta - \lambda = p(ap^{n-1}\alpha - \sigma). \quad (4.6.10)$$

Since η is dominant and λ is restricted, taking the inner product with any simple root β yields

$$-(p-1) \leq \langle\eta - \lambda, \beta^\vee\rangle = p\langle ap^{n-1}\alpha - \sigma, \beta^\vee\rangle.$$

It follows that $\langle ap^{n-1}\alpha - \sigma, \beta^\vee\rangle \geq 0$ and therefore $ap^{n-1}\alpha - \sigma$ is a dominant weight. On the other hand, because $\eta \uparrow \lambda + \gamma$ we have $\langle\lambda + \gamma, \alpha_0^\vee\rangle \geq \langle\eta, \alpha_0^\vee\rangle$ and therefore

$$p/3 > \langle\gamma, \alpha_0^\vee\rangle = \langle\lambda + \gamma - \lambda, \alpha_0^\vee\rangle \geq \langle\eta - \lambda, \alpha_0^\vee\rangle = p\langle ap^{n-1}\alpha - \sigma, \alpha_0^\vee\rangle \geq 0.$$

This forces $ap^{n-1}\alpha - \sigma = 0$, $\lambda = \eta$ and so $\lambda \uparrow \lambda + \gamma$. Finally it follows from [Jan1, II 5.17] that $a = 1$ and hence $\sigma = p^{n-1}\alpha$.

Next we use [Jan1, II 5.15b] to find the highest weight of $H^1(\lambda - p^n\alpha)$. First we assume that $n \geq 2$. Set

$$q = \langle s_\alpha \cdot (\lambda - p^n\alpha) + \rho, \alpha^\vee\rangle = \langle s_\alpha \cdot \lambda + p^n\alpha, \alpha^\vee\rangle + 1 = 2p^n - \langle\lambda, \alpha^\vee\rangle - 1.$$

The p -adic expansion of $q = \sum_{i=0}^n a_i p^i$ has $a_n = 1$, $a_i = p - 1$, for $1 \leq i \leq n - 1$, and $a_0 = p - \langle\lambda, \alpha^\vee\rangle - 1$. By [Jan1, II 5.15b] the highest weight of $H^1(\lambda - p^n\alpha)$ is the largest dominant weight of the form $\mu = \lambda - p^n\alpha + \sum_{i=m}^n a_i p^i \alpha$, where m is greater than or equal to the smallest i with $a_i < p - 1$. Assume that $m < n$, then $\mu = \lambda + \sum_{i=m}^{n-1} a_i p^i \alpha$. If G is not of type A_1 we can find a simple root β with $\langle\alpha, \beta^\vee\rangle < 0$. Taking the inner product with β

yields: $\langle \mu, \beta^\vee \rangle \leq \langle \lambda, \beta^\vee \rangle - a_{n-1} p^{n-1} \leq p-1 - (p-1)p^{n-1} = -(p-1)(p^{n-1}-1) < 0$. Therefore μ is not dominant unless $m = n$. We conclude that the highest weight of $H^1(\lambda - p^n \alpha)$ is λ . It follows from $\text{Hom}_G(V(\lambda + \gamma), H^1(\lambda - p^n \alpha)) \neq 0$ and $\lambda \uparrow \lambda + \gamma$ that $\gamma = 0$. We obtain conclusion (a).

If G is of type A_1 then $H^1(\lambda - p^n \alpha) = H^1(\lambda - 2p^n) \cong V(2p^n - \lambda - 2)$ [Jan1, II 5.11]. We will show that $\text{Hom}_G(V(\lambda + \gamma), V(2p^n - \lambda - 2)) \neq 0$ implies $\gamma = 0$ and hence conclusion (a).

Let us assume that $\text{Hom}_G(V(\lambda + \gamma), V(2p^n - \lambda - 2)) \neq 0$ and $\gamma \neq 0$. We have $\langle \lambda + \gamma + \rho, \alpha^\vee \rangle < \frac{4}{3}p < 2p$. It follows from the Linkage Principle that $\lambda + \gamma = 2p - \lambda - 2$. Moreover, $\lambda \leq p - 2$ and $p > 3$. It is sufficient to show that $L(2p - \lambda - 2)$ is not a composition factor of $V(2p^n - \lambda - 2)$ for $n > 1$. The weights λ and -1 are in the closure of the lowest alcove. We apply [Jan1, II 7.17], a corollary to the Translation Principle, to this pair of weights and obtain $[V(2p^n - \lambda - 2) : L(2p - \lambda - 2)]_G = [V(2p^n - 1) : L(2p - 1)]_G$. Since $\dim V(2p^n - 1) = \dim L(2p^n - 1)$, it follows that $V(2p^n - 1)$ is irreducible. Therefore, for $n > 1$, the above multiplicity is zero and we obtain the desired contradiction.

Finally, assume that G is arbitrary and $n = 1$. Again by [Jan1, II 5.15b], we conclude that the maximal weight of $H^1(\lambda - p\alpha)$ is either $\bar{\lambda} = s_\alpha \cdot (\lambda - p\alpha)$, if $\bar{\lambda}$ is dominant, or λ otherwise. Hence, for dominant $\bar{\lambda}$, one obtains $\lambda \uparrow \lambda + \gamma \uparrow \bar{\lambda}$. Now $\bar{\lambda}$ is the reflection of λ across the (α, p) -wall. It follows that $\gamma = 0$ or $\lambda + \gamma = \bar{\lambda}$. For $\gamma \neq 0$ one has, $\langle \bar{\lambda} - \lambda, \alpha^\vee \rangle = \langle \gamma, \alpha^\vee \rangle < p/3$. Hence, $\langle \lambda + \rho, \alpha^\vee \rangle > p - p/6$. One obtains conclusion (b). \square

Corollary . *Let $\lambda \in X_1(T)$ and $\gamma \in X(T)$ with $\lambda + \gamma$ dominant and $\langle \gamma, \beta^\vee \rangle < p/3$ for any root β . If $\text{Ext}_{B_1}^1(V(\lambda + \gamma), \lambda) \neq 0$, then one of the following holds:*

- (a) $\gamma \neq 0$ and $\lambda + \gamma$ is the image of λ after reflection across the (α, p) -wall, where α denotes a simple root,
- (b) the root system of G is of type A_1 or C_n , and
 - (i) $\langle \lambda, \alpha_n^\vee \rangle = \frac{p-2-c}{2}$, where α_n is the unique long simple root and c is an integer with $0 \leq |c| < p/3$,
 - (ii) $\gamma = \left(\frac{p}{2} - \langle \lambda + \rho, \alpha_n^\vee \rangle\right) \alpha_n$.

Proof. The proof presented here generalizes Andersen's argument that can be found in [Jan1, II 12.3-12.5]. If the B -module $\text{Ext}_{B_1}^1(V(\lambda + \gamma), \lambda) \neq 0$, then there exists a weight σ such that

$$\text{Hom}_{B/B_1}(p\sigma, \text{Ext}_{B_1}^1(V(\lambda + \gamma), \lambda)) \neq 0.$$

The spectral sequence

$$E_2^{i,j} = \text{Ext}_{B/B_1}^i(p\sigma, \text{Ext}_{B_1}^j(V(\lambda + \gamma), \lambda)) \Rightarrow \text{Ext}_B^{i+j}(V(\lambda + \gamma), \lambda - p\sigma)$$

yields the following five term exact sequence

$$\begin{aligned} 0 &\rightarrow \text{Ext}_{B/B_1}^1(p\sigma, \text{Hom}_{B_1}(V(\lambda + \gamma), \lambda)) \rightarrow \text{Ext}_B^1(V(\lambda + \gamma), \lambda - p\sigma) \\ &\rightarrow \text{Hom}_{B/B_1}(p\sigma, \text{Ext}_{B_1}^1(V(\lambda + \gamma), \lambda)) \rightarrow \text{Ext}_{B/B_1}^2(p\sigma, \text{Hom}_{B_1}(V(\lambda + \gamma), \lambda)). \end{aligned}$$

We will distinguish two cases. First suppose that $\gamma \neq 0$. It follows from Lemma 4.4a that the terms

$$E_2^{i,0} = \text{Ext}_{B/B_1}^i(p\sigma, \text{Hom}_{B_1}(V(\lambda + \gamma), \lambda)) = 0.$$

We obtain the isomorphism

$$\mathrm{Ext}_{B/B_1}^1(V(\lambda + \gamma), \lambda - p\sigma) \cong \mathrm{Hom}_{B/B_1}(p\sigma, \mathrm{Ext}_{B_1}^1(V(\lambda + \gamma), \lambda)).$$

Now the assertion follows from the previous proposition.

Now suppose that $\gamma = 0$. We will show that either $\mathrm{Hom}_{B/B_1}(p\sigma, \mathrm{Ext}_{B_1}^1(V(\lambda), \lambda))$ vanishes or we are in case (b). The argument is virtually identical to Andersen's. By Lemma 4.4a, we have

$$E_2^{i,0} = \mathrm{Ext}_{B/B_1}^i(p\sigma, \mathrm{Hom}_{B_1}(V(\lambda), \lambda)) \cong \mathrm{Ext}_{B/B_1}^i(p\sigma, k) \cong H^i(B, -\sigma).$$

The five term exact sequence becomes

$$\begin{aligned} 0 \rightarrow H^1(B, -\sigma) &\rightarrow \mathrm{Ext}_{B/B_1}^1(V(\lambda), \lambda - p\sigma) \\ &\rightarrow \mathrm{Hom}_{B/B_1}(p\sigma, \mathrm{Ext}_{B_1}^1(V(\lambda), \lambda)) \rightarrow H^2(B, -\sigma). \end{aligned}$$

Compare this to [Jan1, II 12.3(5)].

We first show that either we are in case (b) or $\mathrm{Hom}_{B/B_1}(p\sigma, \mathrm{Ext}_{B_1}^1(V(\lambda), \lambda))$ vanishes if $H^2(B, -\sigma)$ vanishes. Indeed, if $\mathrm{Ext}_{B/B_1}^1(V(\lambda), \lambda - p\sigma) = 0$, then the latter claim clearly holds. On the other hand, suppose $\mathrm{Ext}_{B/B_1}^1(V(\lambda), \lambda - p\sigma) \neq 0$. We apply the preceding proposition and conclude from $\gamma = 0$ that either we are in case (b) (correspondingly case (c) of the proposition) as desired or $\sigma = p^m\alpha$ for some simple root α . We continue in the latter case. As noted in the proof of the proposition, we have $\mathrm{Ext}_{B/B_1}^1(V(\lambda), \lambda - p^{m+1}\alpha) \cong \mathrm{Hom}_B(V(\lambda), H^1(\lambda - p^{m+1}\alpha))$. Moreover, the proof shows that the G -socle of $H^1(\lambda - p^{m+1}\alpha)$ is $L(\lambda)$. Therefore, $\mathrm{Ext}_{B/B_1}^1(V(\lambda), \lambda - p^{m+1}\alpha) \cong k$. We also have $H^1(B, -p^m\alpha) \cong k$ by [Jan1, II 5.20]. Hence, $E_2^{1,0} \cong H^1(B, -p^{m+1}\alpha) \cong \mathrm{Ext}_{B/B_1}^1(V(\lambda), \lambda - p^{m+1}\alpha) \cong E^1$ and so $E_2^{0,1} \hookrightarrow E_2^{2,0}$ as desired.

Finally, for a weight $p\sigma$ of $\mathrm{Ext}_{B_1}^1(V(\lambda), \lambda)$, we show that $H^2(B, -\sigma) = 0$. Let $Z_1(\lambda) = \mathrm{coind}_{B^+}^{G_1B^+} \lambda$ as defined in [Jan1, II 9.1]. First observe that

$$\mathrm{Hom}_{G_1B^+}(Z_1(\lambda), V(\lambda)) \cong \mathrm{Hom}_{B^+}(\lambda, V(\lambda)) \neq 0.$$

Therefore, there exists a non-zero G_1B^+ -homomorphism $\phi : Z_1(\lambda) \rightarrow V(\lambda)$. Since $\lambda \in X_1(T)$, $V(\lambda)$ has simple head $L(\lambda)$ as a G_1B^+ -module. The G_1B^+ -head of $Z_1(\lambda)$ is also $L(\lambda)$ and all composition factors in $V(\lambda)$ have weights less than or equal to λ , so ϕ must be surjective. We obtain the exact sequence of G_1B^+ -modules

$$0 \rightarrow M \rightarrow Z_1(\lambda) \rightarrow V(\lambda) \rightarrow 0$$

(compare to [Jan1, II 12.3(3)]). The projectivity of $Z_1(\lambda)$ as a B_1 -module results in an isomorphism of T -modules

$$\mathrm{Ext}_{B_1}^1(V(\lambda), \lambda) \cong \mathrm{Hom}_{B_1}(M, \lambda).$$

It follows for any weight $p\sigma$ of $\mathrm{Ext}_{B_1}^1(V(\lambda), \lambda)$ that $\lambda - p\sigma$ is a weight of M and hence a weight of $Z_1(\lambda)$. As in [Jan1, II 12.4(b)] we conclude that $(p-1)\rho - p\sigma$ is a weight of the Steinberg module. We apply [Jan1, II 12.5] to conclude that $H^2(B, -\sigma) = 0$. \square

4.7 The following proposition reduces possible self-extensions of $L(\lambda)$ to two cases. The first one will be ruled out with the isomorphism in Proposition 4.8.

Proposition . *Let $\lambda, \nu \in X_1(T)$ with $\langle \nu, \alpha_0^\vee \rangle < p/3$. If there exists a weight σ such that $\text{Hom}_{B/B_1}(p\sigma, \text{Ext}_{B_1}^1(V(\lambda), \lambda \otimes H^0(\nu))) \neq 0$, then one of the following hold:*

- (a) σ is equal to a simple root α , $\langle \lambda + \rho, \alpha^\vee \rangle > p - p/6$, and the weight $\bar{\lambda}$, which denotes the image of λ after reflection across the (α, p) -wall, is dominant.
- (b) the underlying root system of G is of type A_1 or C_n and
 - (i) $\sigma = \frac{\alpha_n}{2}$, where α_n is the unique long (last) simple root,
 - (ii) $\langle \lambda, \alpha_n^\vee \rangle = \frac{p-2-c}{2}$, where c is an integer and $0 \leq |c| < p/3$.

Proof. We have $\text{Ext}_{B_1}^1(V(\lambda), \lambda \otimes H^0(\nu)) \cong \text{Ext}_{B_1}^1(V(\lambda) \otimes V(-w_0\nu), \lambda)$. The module $V(\lambda) \otimes V(-w_0\nu)$ has a Weyl filtration [Mat] with factors whose weights are contained in the set $J = \{\mu \in X(T)_+ \mid \text{Hom}_G(V(\lambda) \otimes V(-w_0\nu), H^0(\mu)) \neq 0\}$. Choose a total ordering $\mu_1, \mu_1, \dots, \mu_n$ on J such that $\mu_i > \mu_j$ implies $i < j$. By dualizing the “good filtration” argument in [Jan1, II 4.16], we construct a Weyl filtration $0 = V_0 \subset V_1 \subset V_2 \dots \subset V_n = V$ of $V = V(\lambda) \otimes V(-w_0\nu)$ by setting V_i equal to the kernel of the projection of V_{i+1} onto $V(\mu_{i+1})$. Set $m = \max(\{i \mid \mu_i > \lambda\} \cup \{0\})$. Then the submodule $M = V_m$ has a Weyl filtration whose factors $V(\mu_i)$ satisfy $\mu_i \not\leq \lambda$. The quotient $Q = V/V_m$ has a Weyl filtration whose factors $V(\mu_i)$ satisfy $\mu_i \not\leq \lambda$.

The short exact sequence $0 \rightarrow M \rightarrow V \rightarrow Q \rightarrow 0$ induces the long exact sequence

$$\dots \rightarrow \text{Ext}_{B_1}^1(Q, \lambda) \rightarrow \text{Ext}_{B_1}^1(V, \lambda) \rightarrow \text{Ext}_{B_1}^1(M, \lambda) \rightarrow \dots$$

If $\text{Ext}_{B_1}^1(Q, \lambda) \neq 0$ then there exists a $V(\mu_i)$ in the filtration of Q with $\text{Ext}_{B_1}^1(V(\mu_i), \lambda) \neq 0$. By Corollary 4.6 we can either conclude (b) or $\mu_i > \lambda$ which contradicts how Q was constructed. Therefore, we may now assume that $\text{Ext}_{B_1}^1(Q, \lambda) = 0$. The exact sequence becomes

$$0 \rightarrow \text{Ext}_{B_1}^1(V, \lambda) \rightarrow \text{Ext}_{B_1}^1(M, \lambda) \rightarrow \dots$$

Using left exactness of the functor $\text{Hom}_{B/B_1}(p\sigma, -)$, there is an injection:

$$0 \rightarrow \text{Hom}_{B/B_1}(p\sigma, \text{Ext}_{B_1}^1(V, \lambda)) \hookrightarrow \text{Hom}_{B/B_1}(p\sigma, \text{Ext}_{B_1}^1(M, \lambda)).$$

Hence, $\text{Hom}_{B/B_1}(p\sigma, \text{Ext}_{B_1}^1(M, \lambda)) \neq 0$.

For $i = 0, 1, \dots, m-1$, we have $V_{i+1}/V_i \cong V(\mu_{i+1})$. Consider the short exact sequences

$$0 \rightarrow V(\mu_{i+1}) \rightarrow M/V_i \rightarrow M/V_{i+1} \rightarrow 0.$$

The construction of M allows only factors $V(\mu)$ with $\mu \neq \lambda$. Hence, it follows from Lemma 4.4a that the long exact sequence

$$\dots \rightarrow \text{Hom}_{B_1}(V(\mu_{i+1}), \lambda) \rightarrow \text{Ext}_{B_1}^1(M/V_{i+1}, \lambda) \rightarrow \text{Ext}_{B_1}^1(M/V_i, \lambda) \rightarrow \text{Ext}_{B_1}^1(V(\mu_{i+1}), \lambda) \rightarrow \dots$$

becomes

$$0 \rightarrow \text{Ext}_{B_1}^1(M/V_{i+1}, \lambda) \rightarrow \text{Ext}_{B_1}^1(M/V_i, \lambda) \rightarrow \text{Ext}_{B_1}^1(V(\mu_{i+1}), \lambda) \rightarrow \dots$$

Again by using the left exactness of the functor $\text{Hom}_{B/B_1}(p\sigma, -)$, we obtain exact sequences for all $i = 0, \dots, m-1$:

$$\begin{aligned} 0 \rightarrow \text{Hom}_{B/B_1}(p\sigma, \text{Ext}_{B_1}^1(M/V_{i+1}, \lambda)) &\rightarrow \text{Hom}_{B/B_1}(p\sigma, \text{Ext}_{B_1}^1(M/V_i, \lambda)) \rightarrow \\ &\text{Hom}_{B/B_1}(p\sigma, \text{Ext}_{B_1}^1(V(\mu_{i+1}), \lambda)) \rightarrow \dots \end{aligned}$$

Since

$$0 \neq \text{Hom}_{B/B_1}(p\sigma, \text{Ext}_{B_1}^1(M, \lambda)) = \text{Hom}_{B/B_1}(p\sigma, \text{Ext}_{B_1}^1(M/V_0, \lambda)),$$

while clearly $\text{Hom}_{B/B_1}(p\sigma, \text{Ext}_{B_1}^1(M/V_m, \lambda)) = 0$, there exists a $\mu \neq \lambda$ with

$$\text{Hom}_{B/B_1}(p\sigma, \text{Ext}_{B_1}^1(V(\mu), \lambda)) \neq 0.$$

μ is of the form $\lambda + \gamma$ with $\gamma \neq 0$. The size of ν forces $\langle \gamma, \beta^\vee \rangle < p/3$ for all roots β . We repeat the argument of the $\gamma \neq 0$ case in the proof of Corollary 4.6 to conclude that

$$\text{Ext}_B^1(V(\lambda + \gamma), \lambda - p\sigma) \cong \text{Hom}_{B/B_1}(p\sigma, \text{Ext}_{B_1}^1(V(\lambda + \gamma), \lambda)) \neq 0.$$

The conclusion follows now from Proposition 4.6. □

4.8 The following isomorphism will allow us to deal with weights λ close to an upper wall of the restricted region.

Proposition . *Let $\lambda, \nu \in X_1(T)$ with $\langle \nu, \alpha_0^\vee \rangle < p/3$, and $\alpha \in \Delta$. If $\langle \lambda + \rho, \alpha^\vee \rangle > p - p/6$ and the weight $\bar{\lambda}$, which denotes the image of λ after reflection across the (α, p) -wall, is dominant, then*

$$\text{Ext}_B^1(V(\lambda), (\lambda - p\alpha) \otimes H^0(\nu)) \cong \text{Hom}_B(V(\lambda), \lambda \otimes H^0(\nu)).$$

Proof. First observe that since $\lambda - p\alpha \notin X(T)_+$, the induction spectral sequence as used in the proof of Proposition 4.6 gives an isomorphism

$$\text{Ext}_B^1(V(\lambda), (\lambda - p\alpha) \otimes H^0(\nu)) \cong \text{Hom}_G(V(\lambda), R^1 \text{ind}_B^G(\lambda - p\alpha) \otimes H^0(\nu)).$$

Let $P(\alpha)$ and $L(\alpha)$ denote parabolic and Levi subgroups that correspond to the simple root α . This convention should not be confused with our notation for simple G -modules. The reader should be able to avoid any confusion from the context in which the notation is used. According to [Jan1, II 5.13], $\text{ind}_{P(\alpha)}^G R^1 \text{ind}_B^{P(\alpha)}(\lambda - p\alpha) \cong R^1 \text{ind}_B^G(\lambda - p\alpha)$. Using Frobenius reciprocity, we then have

$$\text{Ext}_B^1(V(\lambda), (\lambda - p\alpha) \otimes H^0(\nu)) \cong \text{Hom}_{P(\alpha)}(V(\lambda), R^1 \text{ind}_B^{P(\alpha)}(\lambda - p\alpha) \otimes H^0(\nu)).$$

On the other hand, applying Frobenius reciprocity twice, we have

$$\begin{aligned} \text{Hom}_B(V(\lambda), \lambda \otimes H^0(\nu)) &\cong \text{Hom}_G(V(\lambda), H^0(\lambda) \otimes H^0(\nu)) \\ &\cong \text{Hom}_{P(\alpha)}(V(\lambda), \text{ind}_B^{P(\alpha)} \lambda \otimes H^0(\nu)) \end{aligned}$$

Thus we are reduced to proving that

$$\text{Hom}_{P(\alpha)}(V(\lambda), R^1 \text{ind}_B^{P(\alpha)}(\lambda - p\alpha) \otimes H^0(\nu)) \cong \text{Hom}_{P(\alpha)}(V(\lambda), \text{ind}_B^{P(\alpha)}(\lambda) \otimes H^0(\nu)). \quad (4.8.1)$$

To show this isomorphism, we investigate certain $P(\alpha)$ -modules of the form $R^i \text{ind}_B^{P(\alpha)}(\sigma)$ for $\sigma \in X(T)$. The unipotent radical of $P(\alpha)$ acts trivially on any $R^i \text{ind}_B^{P(\alpha)}(\sigma)$ by [Jan1, II 4.6(a), I 6.11]. Hence, the structure of an $R^i \text{ind}_B^{P(\alpha)}(\sigma)$ can be understood by considering its structure as an $L(\alpha)$ -module. We denote the simple $L(\alpha)$ -modules by $L_\alpha(\sigma)$. This is also a simple $P(\alpha)$ -module by letting the unipotent radical act trivially.

As an $L(\alpha)$ -module, $R^1 \text{ind}_B^{P(\alpha)}(\lambda - p\alpha)$ is just the Weyl module with highest weight $\bar{\lambda}$. It has a composition series of length two. Its socle is $L_\alpha(\lambda)$ and its head is $L_\alpha(\bar{\lambda})$. Hence, there is a short exact sequence of $P(\alpha)$ -modules

$$0 \rightarrow L_\alpha(\lambda) \rightarrow R^1 \text{ind}_B^{P(\alpha)}(\lambda - p\alpha) \rightarrow L_\alpha(\bar{\lambda}) \rightarrow 0. \quad (4.8.2)$$

For any simple root $\beta \neq \alpha$, we have $0 \leq \langle \bar{\lambda}, \beta^\vee \rangle \leq \langle \lambda, \beta^\vee \rangle \leq p-1$. Therefore we can write $\bar{\lambda} = \bar{\lambda}_0 + p\omega_\alpha$, where $\bar{\lambda}_0 \in X_1(T)$ and ω_α denotes the fundamental weight corresponding to α . By Steinberg's tensor product theorem, $L_\alpha(\bar{\lambda}) \cong L_\alpha(\bar{\lambda}_0) \otimes L_\alpha(\omega_\alpha)^{(1)}$. Each of the weights $\lambda, \bar{\lambda}_0$, and ω_α is p -restricted. Therefore, as $P(\alpha)$ -modules, $L_\alpha(\lambda) \cong \text{ind}_B^{P(\alpha)}(\lambda)$ and $L_\alpha(\bar{\lambda}) \cong \text{ind}_B^{P(\alpha)}(\bar{\lambda}_0) \otimes (\text{ind}_B^{P(\alpha)}(\omega_\alpha))^{(1)}$. Hence the short exact sequence (4.8.2) of $P(\alpha)$ -modules can be expressed as

$$0 \rightarrow \text{ind}_B^{P(\alpha)}(\lambda) \rightarrow R^1 \text{ind}_B^{P(\alpha)}(\lambda - p\alpha) \rightarrow \text{ind}_B^{P(\alpha)}(\bar{\lambda}_0) \otimes (\text{ind}_B^{P(\alpha)}(\omega_\alpha))^{(1)} \rightarrow 0.$$

From this, one obtains the exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}_{P(\alpha)}(V(\lambda), \text{ind}_B^{P(\alpha)}(\lambda) \otimes H^0(\nu)) &\rightarrow \text{Hom}_{P(\alpha)}(V(\lambda), R^1 \text{ind}_B^{P(\alpha)}(\lambda - p\alpha) \otimes H^0(\nu)) \\ &\rightarrow \text{Hom}_{P(\alpha)}(V(\lambda), \text{ind}_B^{P(\alpha)}(\bar{\lambda}_0) \otimes (\text{ind}_B^{P(\alpha)}(\omega_\alpha))^{(1)} \otimes H^0(\nu)) \rightarrow \dots \end{aligned}$$

Now it follows from (4.8.1) that it suffices to show that

$$\text{Hom}_{P(\alpha)}(V(\lambda), \text{ind}_B^{P(\alpha)}(\bar{\lambda}_0) \otimes (\text{ind}_B^{P(\alpha)}(\omega_\alpha))^{(1)} \otimes H^0(\nu)) = 0. \quad (4.8.3)$$

As above, the composition factors of $N = \text{ind}_B^{P(\alpha)}(\bar{\lambda}_0) \otimes H^0(\nu) \cong L_\alpha(\bar{\lambda}_0) \otimes H^0(\nu)$ as a $P(\alpha)$ -module are easily obtained by considering N as an $L(\alpha)$ -module. They have the form $L_\alpha(\sigma)$ where the unipotent radical of $P(\alpha)$ acts trivially. If δ is a weight of $\text{ind}_B^{P(\alpha)}(\bar{\lambda}_0)$ and τ is a weight of $H^0(\nu)$ then

$$\langle \delta + \tau, \alpha^\vee \rangle \leq \langle \bar{\lambda}_0, \alpha^\vee \rangle + \langle \nu, \alpha_0^\vee \rangle < \frac{p}{6} + \frac{p}{3} < p.$$

Consequently, all composition factors of N have the form $L_\alpha(\sigma)$ where σ is a p -restricted weight (for $L(\alpha)$). By using an inductive argument on a composition series of N , it now suffices to prove that

$$\text{Hom}_{P(\alpha)}(V(\lambda), L_\alpha(\sigma) \otimes (\text{ind}_B^{P(\alpha)}(\omega_\alpha))^{(1)}) = 0$$

for all σ that are p -restricted relative to $L(\alpha)$. Since $L_\alpha(\sigma) \otimes (\text{ind}_B^{P(\alpha)}(\omega_\alpha))^{(1)} \cong L_\alpha(\sigma) \otimes L_\alpha(\omega_\alpha)^{(1)}$ is a simple $P(\alpha)$ -module, there is a monomorphism

$$L_\alpha(\sigma) \otimes L_\alpha(\omega_\alpha)^{(1)} \hookrightarrow \text{ind}_B^{P(\alpha)}(\sigma + p\omega_\alpha).$$

This induces another monomorphism

$$\text{Hom}_{P(\alpha)}(V(\lambda), L_\alpha(\sigma) \otimes L_\alpha(\omega_\alpha)^{(1)}) \hookrightarrow \text{Hom}_{P(\alpha)}(V(\lambda), \text{ind}_B^{P(\alpha)}(\sigma + p\omega_\alpha)).$$

Finally, by Frobenius reciprocity and the fact that $\sigma + p\omega_\alpha \notin X_1(T)$ whereas $\lambda \in X_1(T)$, we have

$$\text{Hom}_{P(\alpha)}(V(\lambda), \text{ind}_B^{P(\alpha)}(\sigma + p\omega_\alpha)) \cong \text{Hom}_G(V(\lambda), H^0(\sigma + p\omega_\alpha)) = 0.$$

□

4.9 Vanishing of B_1 -cohomology The following theorem serves the same purpose as [Jan1, II 12.6] in Andersen's proof of the vanishing of self-extensions for Frobenius kernels. Indeed, together with Theorem 2.5 and Lemma 4.5, this proves Theorem 4.2.

Theorem . *Let $\lambda, \nu \in X_1(T)$ with $\langle \nu, \alpha_0^\vee \rangle < p/3$. If $\text{Ext}_{B_1}^1(V(\lambda), \lambda \otimes H^0(\nu)) \neq 0$, then*

- (i) *G has underlying root system of type A_1 or C_n and*
- (ii) *$\langle \lambda, \alpha_n^\vee \rangle = \frac{p-2-c}{2}$, where α_n is the unique long simple root and c is an integer with $0 \leq |c| < p/3$.*

Proof. There exists a weight σ such that $\text{Hom}_{B/B_1}(p\sigma, \text{Ext}_{B_1}^1(V(\lambda), \lambda \otimes H^0(\nu))) \neq 0$. Either (a) or (b) in Proposition 4.7 must hold. It suffices to eliminate case (a). Assuming case (a), we have $\sigma = \alpha$, a simple root.

Again we will use a modification of Andersen's argument given in [Jan1, II 12.3-12.5]. There exists a spectral sequence

$$\text{Ext}_{B/B_1}^i(p\alpha, \text{Ext}_{B_1}^j(V(\lambda), \lambda \otimes H^0(\nu))) \Rightarrow \text{Ext}_B^{i+j}(V(\lambda), (\lambda - p\alpha) \otimes H^0(\nu)).$$

The beginning of the corresponding five-term exact sequence is

$$\begin{aligned} 0 &\rightarrow \text{Ext}_{B/B_1}^1(p\alpha, \text{Hom}_{B_1}(V(\lambda), \lambda \otimes H^0(\nu))) \rightarrow \text{Ext}_B^1(V(\lambda), (\lambda - p\alpha) \otimes H^0(\nu)) \\ &\rightarrow \text{Hom}_{B/B_1}(p\alpha, \text{Ext}_{B_1}^1(V(\lambda), \lambda \otimes H^0(\nu))) \rightarrow \text{Ext}_{B/B_1}^2(p\alpha, \text{Hom}_{B_1}(V(\lambda), \lambda \otimes H^0(\nu))). \end{aligned}$$

By Lemma 4.4b, $\text{Hom}_{B_1}(V(\lambda), \lambda \otimes H^0(\nu))$ is a direct sum of trivial modules. Then

$$\begin{aligned} \text{Ext}_{B/B_1}^1(p\alpha, \text{Hom}_{B_1}(V(\lambda), \lambda \otimes H^0(\nu))) &\cong H^1(B/B_1, -p\alpha) \otimes \text{Hom}_{B_1}(V(\lambda), \lambda \otimes H^0(\nu)) \\ \text{Ext}_{B/B_1}^2(p\alpha, \text{Hom}_{B_1}(V(\lambda), \lambda \otimes H^0(\nu))) &\cong H^2(B/B_1, -p\alpha) \otimes \text{Hom}_{B_1}(V(\lambda), \lambda \otimes H^0(\nu)). \end{aligned}$$

By [Jan1, II 5.20], $H^1(B/B_1, -p\alpha) \cong k$ and by [Jan1, II 12.5], $H^2(B/B_1, -p\alpha) = 0$. Therefore, the exact sequence becomes

$$\begin{aligned} 0 \rightarrow \text{Hom}_{B_1}(V(\lambda), \lambda \otimes H^0(\nu)) &\rightarrow \text{Ext}_B^1(V(\lambda), (\lambda - p\alpha) \otimes H^0(\nu)) \\ &\rightarrow \text{Hom}_{B/B_1}(p\alpha, \text{Ext}_{B_1}^1(V(\lambda), \lambda \otimes H^0(\nu))) \rightarrow 0. \end{aligned}$$

By Proposition 4.8, it follows that the first map is an isomorphism, thus $\text{Hom}_{B/B_1}(p\alpha, \text{Ext}_{B_1}^1(V(\lambda), \lambda \otimes L(\nu))) = 0$ which is a contradiction. \square

The following Corollary together with Theorem 2.5 implies Theorem 4.2.

Corollary . *Let $p \geq 3(h-1)$, $\lambda \in X_1(T)$, and $\nu_1, \nu_2 \in \Gamma$ with $\nu_1 \neq 0$. If either*

- (a) *G does not have underlying root system of type A_1 or C_n or*
- (b) *$\langle \lambda, \alpha_n^\vee \rangle \neq \frac{p-2-c}{2}$, where α_n is the unique long simple root and c is odd with $0 < |c| \leq h-1$,*

then $\text{Ext}_G^1(L(\lambda) \otimes L(\nu_1)^{(1)}, L(\lambda) \otimes L(\nu_2)) = 0$.

Proof. The size of p forces all elements of Γ to be in the lowest alcove, hence $L(\nu_i) = V(\nu_i) = H^0(\nu_i)$. Moreover, $\langle \nu_i, \alpha_0^\vee \rangle < p/3$, unless $p = 3$ and G is of type A_1 , in which case the assertion can be verified directly. The claim now follows from Theorem 4.9 and Lemma 4.5. \square

5 Completely reducible modules

5.1 In 1985, S.D. Smith [S] studied the reducibility of certain $G(\mathbb{F}_q)$ -modules and posed the following question.

(5.1.1) ([S, Question 5]) If a finite-dimensional $G(\mathbb{F}_q)$ -module M has a composition series with all composition factors being isomorphic must M be completely reducible?

Some remarks must be made at this point. First, throughout this paper a $G(\mathbb{F}_q)$ -module has properly designated a $kG(\mathbb{F}_q)$ -module. In [S], Smith considers $\mathbb{F}_qG(\mathbb{F}_q)$ -modules. However, an $\mathbb{F}_qG(\mathbb{F}_q)$ -module V is completely reducible if and only if $V \otimes_{\mathbb{F}_q} k$ is completely reducible over $kG(\mathbb{F}_q)$. Hence, for consistency, the results of this section will be stated over the field k . Secondly, Smith originally posed the question only for certain “highest weight” modules. This assumption is not necessary for the succeeding results.

Smith answers (5.1.1) affirmatively ([S, Proposition 4]) under a further assumption that the highest weight of the “unique” composition factor is minimal. On the other hand, he notes that the answer is no in general as there exists an indecomposable $SL_2(\mathbb{F}_p)$ -module with two isomorphic composition factors. Indeed, this is related to our exclusion of certain weights in Theorem 4.2. We can use the above results on self-extensions to show that the answer to (5.1.1) is yes in most cases when the prime is sufficiently large.

Theorem (A). *Let $p \geq 3(h-1)$ and $r \geq 2$. If M is a finite-dimensional $G(\mathbb{F}_q)$ -module with all composition factors being isomorphic, then M is completely reducible.*

Theorem (B). *Let $p \geq 3(h-1)$. Suppose M is a finite-dimensional $G(\mathbb{F}_p)$ -module with all composition factors being isomorphic to $L(\lambda)$ for some $\lambda \in X_1(T)$. If the underlying root system of G is of type A_1 or C_n , assume further that, $\langle \lambda, \alpha_n^\vee \rangle \neq \frac{p-2-c}{2}$, where α_n is the unique long simple root and c is odd with $0 < |c| \leq h-1$. Then M is completely reducible.*

5.2 The vanishing of extensions between simple modules for a ring R is equivalent to the ring being semisimple. Furthermore, for a ring R and R -module M , the condition that $\text{Ext}_R^1(L, L') = 0$ for all (not necessarily distinct) composition factors L and L' of M implies that M is completely reducible. Corollary 3.3 along with Theorem 3.4 can be used to deduce a somewhat more general result about complete reducibility of $G(\mathbb{F}_q)$ -modules when $r \geq 2$.

Theorem . *Let $p \geq 3(h-1)$ and $r \geq 2$. Suppose that M is a finite-dimensional $G(\mathbb{F}_q)$ -module such that $\text{Ext}_G^1(L(\lambda), L(\mu)) = 0$ for all composition factors $L(\lambda)$ and $L(\mu)$ of M and for $\lambda \neq \mu$ either*

(i) $\lambda_{r-1} = \mu_{r-1}$ or

(ii) *there exists a root α such that $|\langle \hat{\lambda} - \hat{\mu}, \alpha^\vee \rangle| > h-1$.*

Then M is completely reducible.

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