

Second Cohomology Groups for Frobenius kernels and related structures

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Abstract

Let G be a simple simply connected algebraic group over an algebraically closed field k of characteristic p for an odd prime p . Let B be a Borel subgroup of G and U be its unipotent radical. In this paper, we determine the second cohomology groups of B and its Frobenius kernels for all simple B -modules. We also consider the standard induced modules obtained by inducing a simple B -module to G and compute all second cohomology groups of the Frobenius kernels of G for these induced modules. Also included is a calculation of the second ordinary Lie algebra cohomology group of $\text{Lie}(U)$ with coefficients in k .

Key words: Algebraic groups, Lie algebra cohomology, Frobenius kernels
1991 MSC: 17B50, 17B56, 20G05, 20G10

1 Introduction

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¹ Supported in part by NSA grant MDA904-02-1-0078 and NSF grant DMS-0400558

² Supported in part by NSF grants DMS-0102225 and DMS-0400548

1.1 Let G be a simple simply connected affine algebraic group defined over \mathbb{F}_p and k be an algebraically closed field of characteristic $p > 0$. Let $F : G \rightarrow G$ denote the Frobenius morphism. The scheme-theoretic kernel of F is the first Frobenius kernel G_1 . It is well known that the representation theory of G_1 is equivalent to the restricted representation theory of the (restricted) Lie algebra $\mathfrak{g} = \text{Lie}(G)$. More generally, for a positive integer r , higher Frobenius kernels G_r can be defined by taking the kernel of the iteration of F with itself r times.

For a dominant integral weight λ , the space of global sections $\mathcal{H}^0(G/B, \mathcal{L}(\lambda))$ of the line bundle $\mathcal{L}(\lambda)$ over the flag variety G/B is an object of central importance in the representation theory of G . These sections are G -modules and the socles of these modules constitute the collection of isomorphism classes of finite-dimensional simple G -modules. The G -module $\mathcal{H}^0(G/B, \mathcal{L}(\lambda))$ can be identified with $H^0(\lambda) = \text{ind}_B^G \lambda$ where λ is regarded as a one-dimensional module for the Borel subgroup B [Jan1, II.4.2].

In 1984, Andersen and Jantzen [AJ] made the following fundamental computation of G_1 -cohomology groups with coefficients in $H^0(\lambda)$. This generalized the result for the trivial module $k = H^0(0)$ obtained by Friedlander and Parshall [FP2]. Andersen and Jantzen's results required some special assumptions for certain exceptional groups. Recently, Kumar, Lauritzen and Thomsen [KLT] employed Frobenius splittings to prove the result for all types:

Theorem *Suppose $p > h$, $\mu \in X(T)_+$, $w \in W$, and $w \cdot 0 + p\mu \in X(T)_+$. Then*

$$H^i(G_1, H^0(w \cdot 0 + p\mu))^{(-1)} \cong \begin{cases} \text{ind}_B^G \left(S^{(i-\ell(w))/2}(\mathfrak{u}^*) \otimes \mu \right) & \text{if } i - \ell(w) \text{ is even,} \\ 0 & \text{otherwise} \end{cases}$$

where $\mathfrak{u} = \text{Lie}(U)$ for the unipotent radical U of the Borel B .

Note that $H^i(G_1, H^0(\lambda)) = 0$ for any weight λ which is either not dominant nor of the form $w \cdot 0 + p\mu$. For a G -module M , we denote by $M^{(r)}$ the G -module obtained by composing the G -action with $F^r : G \rightarrow G$. Observe that G_r acts trivially on $M^{(r)}$. Conversely, if N is a G -module on which G_r -acts trivially, then there is a unique G -module M with $N = M^{(r)}$. We denote the module M by $N^{(-r)}$.

In this paper, our goal is to extend this theorem for $i = 2$ in two directions. First, we wish to study the G_1 -cohomology for small primes. More generally, we will consider G_r -cohomology for arbitrary $r \geq 1$. In the first direction, Andersen and Jantzen [AJ] considered some small prime cases ($p = h$, $p = h - 1$, and type G_2 for $p = 3$). A first step toward a complete description was provided by work of Jantzen [Jan2]. Jantzen computed $H^1(G_1, H^0(\lambda))$ for all λ and all primes, and his computations showed that there is a generic answer

for $p > 3$ which is much less restrictive than the $p > h$ condition. In recent work [BNP], the authors extended his results to compute $H^1(G_r, H^0(\lambda))$ for all p, λ , and r . The main goal of this paper is to demonstrate how to compute $H^2(G_r, H^0(\lambda))$ for all $p \geq 3$, $\lambda \in X(T)_+$ and $r \geq 1$. It is interesting to note that our results do not rely on Frobenius splittings, but rather strongly on our previous calculations for $H^1(G_r, H^0(\lambda))$ (e.g. see Theorem 6.1).

1.2 The strategy we follow consists of a sequence of reduction steps. In the process, several other significant cohomology computations are obtained. First, the computation of G_r -cohomology can be reduced to the computation of B_r -cohomology (see Theorem 6.1):

$$H^2(G_r, H^0(\lambda))^{(-r)} \cong \text{ind}_B^G(H^2(B_r, \lambda)^{(-r)})$$

With the aid of the Lyndon-Hochschild-Serre spectral sequence for $B_1 \subset B_r$, the B_r -cohomology for arbitrary r can be determined from the B_1 -cohomology. The problem is further reduced to the computation of $H^2(U_1, k)$ via the isomorphism (see Section 4.2)

$$H^2(B_1, \lambda) \cong (H^2(U_1, k) \otimes \lambda)^{T_1}.$$

Finally, the problem is reduced further to a computation of the ordinary Lie algebra cohomology $H^2(\mathfrak{u}, k)$ thanks to a spectral sequence of Friedlander and Parshall (see Section 4.1) for $p \geq 3$ which relates $H^2(U_1, k)$ to $H^2(\mathfrak{u}, k)$. The computation of $H^2(\mathfrak{u}, k)$ is rather subtle and not direct. In order to complete this calculation we need to use basic calculations involving root sums, in addition to employing some information about non-vanishing of the rational cohomology for B , due to Andersen [And], as well as for the Frobenius kernel B_1 .

1.3 The paper is organized in the opposite direction to the above reduction steps. In Section 2, we remind the reader of the definition of ordinary Lie algebra cohomology and present some initial observations about first and second degree cohomology. In Section 3, we investigate conditions involving root sums which play a role in determining \mathfrak{u} -cohomology as well as understanding the connections with U_1 - and B_1 -cohomology. In Section 4, we compute $H^2(\mathfrak{u}, k)$ for $p \geq 3$. In Section 5, we first determine $H^2(B_1, \lambda)$ ($p \geq 3$). This calculation allows us to give a complete answer for $H^2(B_r, \lambda)$ for $\lambda \in X(T)$. Finally, in Section 6, we apply the induction functor ind_B^G to $H^2(B_r, \lambda)$ to make computations of $H^2(G_r, H^0(\lambda))$.

Specifically, we will provide the following second cohomology computations when $p \geq 3$. Our restriction to $p \geq 3$ is due to the aforementioned use of a spectral sequence which holds only for $p \geq 3$ and the large number of special

cases that would arise for example in Section 3 when $p = 2$.

(1.3.1) $H^2(\mathfrak{u}, k)$ - see Theorem 4.4.

(1.3.2) $H^2(B_r, \lambda)$ where $\lambda \in X(T)$ - see Theorem 5.3 and Theorem 5.7.

(1.3.3) $H^2(B, \lambda)$ where $\lambda \in X(T)$ - see Theorem 5.8.

(1.3.4) $H^2(G_r, H^0(\lambda))$ where $\lambda \in X(T)_+$ - see Theorem 6.2 and Theorem 6.3.

Since $H^2(B_r, \lambda) \cong (H^2(U_r, k) \otimes \lambda)^{T_r}$, the complete information on $H^2(B_r, \lambda)$ given below could be used to compute the T -structure of $H^2(U_r, k)$.

The computation of \mathfrak{u} -cohomology (1.3.1) extends work of Kostant [Kos] in characteristic zero and improves work of Friedlander-Parshall [FP3] and Polo-Tilouine [PT] which required p to be at least one less than the Coxeter number. The computation of the B -cohomology (1.3.3) is significant because of its potential connection with computing higher line bundle cohomology groups $\mathcal{H}^i(G/B, \mathcal{L}(\lambda))$ for $i \geq 1$. Further, knowledge about second cohomology groups are important because they provide information about central extensions of the underlying algebraic structures.

1.4 Notation: Throughout this paper, we will follow the standard conventions provided in [Jan1]. Let G be a simple simply connected algebraic group scheme defined and split over the finite field \mathbb{F}_p with p elements, and k be the algebraic closure of \mathbb{F}_p . For $r \geq 1$, let G_r be the r th Frobenius kernel of G . Let T be a maximal split torus and Φ be the root system associated to (G, T) . The positive (resp. negative) roots are Φ^+ (resp. Φ^-), and Δ is the set of simple roots. Let B be a Borel subgroup containing T corresponding to the negative roots and U be the unipotent radical of B . For a given root system of rank n , the simple roots will be denoted by $\alpha_1, \alpha_2, \dots, \alpha_n$. We will adhere to the ordering of the simple roots as given in [Jan2] (following Bourbaki). In particular, for type B_n , α_n denotes the unique short simple root and for type C_n , α_n denotes the unique long simple root. If $\alpha \in \Phi$ and $\alpha = \sum_{i=1}^n m_i \alpha_i$ then the height of α is defined by $\text{ht}(\alpha) := \sum_{i=1}^n m_i$.

Let \mathbb{E} be the Euclidean space associated with Φ . The inner product on \mathbb{E} will be denoted by $\langle \cdot, \cdot \rangle$. Let $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ be the coroot corresponding to $\alpha \in \Phi$. In this case, the fundamental weights (basis dual to $\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_n^\vee$) will be denoted by $\omega_1, \omega_2, \dots, \omega_n$. Let h denote the Coxeter number associated to Φ , and W be the Weyl group. Let $X(T)$ be the integral weight lattice spanned by these fundamental weights. The set of dominant integral weights is denoted by $X(T)_+$ and the set of p^r -restricted weights is $X_r(T)$. Note that W acts naturally on $X(T)$ and also on $X(T)$ via the dot action.

Each character $\lambda \in X(T)_+$ defines a one-dimensional B -module by letting U act trivially. For convenience, we simply denote this module by λ . The induced module $H^0(\lambda) = \text{ind}_B^G \lambda$ has a simple socle which will be denoted $L(\lambda)$. The

set $\{L(\lambda) : \lambda \in X(T)_+\}$ is a set of representatives for the isomorphism classes of simple G -modules.

2 Observations on \mathfrak{u} -cohomology

2.1 We will be interested in the ordinary Lie algebra cohomology of the Lie algebra $\mathfrak{u} = \text{Lie}(U)$. We remind the reader of the definition of the ordinary Lie algebra cohomology of a Lie algebra L over k . The ordinary Lie algebra cohomology $H^i(L, k)$ may be computed as the cohomology of the complex

$$k \xrightarrow{d_0} L^* \xrightarrow{d_1} \Lambda^2(L)^* \xrightarrow{d_2} \Lambda^3(L)^* \rightarrow \dots .$$

The differentials are given as follows: $d_0 = 0$ and $d_1 : L^* \rightarrow \Lambda^2(L)^*$ with

$$(d_1\phi)(x \wedge y) = -\phi([x, y])$$

where $\phi \in L^*$ and $x, y \in L$. For the higher differentials, we identify $\Lambda^n(L)^* \cong \Lambda^n(L^*)$. Then the differentials are determined by the following product rule (see [Jan1, I.9.17]):

$$d_{i+j}(\phi \wedge \psi) = d_i(\phi) \wedge \psi + (-1)^i \phi \wedge d_j(\psi).$$

2.2 First cohomology of \mathfrak{u} : Now let us consider the case when $L = \mathfrak{u} = \text{Lie } U$. Information about $H^2(\mathfrak{u}, k)$ will be used to make our computations of B_r -cohomology. Such computations are also interesting in their own right. A basis for \mathfrak{u} is given by a basis of negative root vectors $\{x_\alpha : \alpha \in \Phi^-\}$. Let $\{\phi_\alpha : \alpha \in \Phi^+\}$ be the dual basis in \mathfrak{u}^* with $\phi_\alpha(x_\beta) = \delta_{-\alpha, \beta}$ for all $\alpha \in \Phi^+$ and $\beta \in \Phi^-$. Since U is normal in B , B acts on \mathfrak{u} by the adjoint action. This action naturally extends to an action on \mathfrak{u}^* . Under this action, ϕ_α has a T -eigenvector with T -weight α . The action of B further extends naturally to any exterior power $\Lambda^i(\mathfrak{u}^*)$ and, since the operation is preserved by the differentials, to the cohomology $H^i(\mathfrak{u}, k)$.

It is well known that the first cohomology (for any Lie algebra) is

$$H^1(\mathfrak{u}, k) = \ker d_1 = (\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}])^*.$$

For large primes, the simple roots give a basis for $H^1(\mathfrak{u}, k)$. Specifically, we recall the following result of Jantzen [Jan2].

Proposition *Assume $p \geq 3$.*

- (a) If $p = 3$, assume further that Φ is not of type G_2 . Then a basis of T -eigenvectors for $H^1(\mathbf{u}, k)$ is $\{\phi_\alpha : \alpha \in \Delta\}$.
- (b) Suppose $p = 3$ and Φ is of type G_2 . Then a basis of T -eigenvectors for $H^1(\mathbf{u}, k)$ is $\{\phi_{\alpha_1}, \phi_{\alpha_2}, \phi_{3\alpha_1+\alpha_2}\}$.

2.3 We now turn to $H^2(\mathbf{u}, k)$. Cohomology classes will be represented by linear combinations of elements of the form $\phi_\alpha \wedge \phi_\beta$ for $\alpha, \beta \in \Phi^+$. We observe the following.

Proposition *Let $w \in W$. Then*

$$\dim_k \Lambda^n(\mathbf{u}^*)_{-w \cdot 0} = \begin{cases} 1 & \text{if } n = l(w) \\ 0 & \text{otherwise.} \end{cases}$$

Let $x_w \in \Lambda^{l(w)}(\mathbf{u}^)$ be an element of weight $-w \cdot 0$. Then x_w represents a cohomology class in $H^{l(w)}(\mathbf{u}, k)$.*

Proof. The first claim is a special case of [FP3, Prop. 2.2] where it was observed that the result originally known to hold in characteristic zero holds also in prime characteristic. To see that x_w is a cohomology class, recall that the differential $d_i : \Lambda^i(\mathbf{u}^*) \rightarrow \Lambda^{i+1}(\mathbf{u}^*)$ (see Section 2.2) preserves the action of T . Since there are no elements of weight $-w \cdot 0$ in $\Lambda^{l(w)+1}(\mathbf{u}^*)$ or $\Lambda^{l(w)-1}(\mathbf{u}^*)$, x_w is necessarily a cocycle and cannot be a coboundary. \square

Over characteristic zero, $H^\bullet(\mathbf{u}, k)$ was computed by Kostant [Kos]. The cohomology classes in the preceding proposition give a basis of T -eigenvectors. In prime characteristic, it is known for $p \geq h - 1$ by work of Friedlander and Parshall [FP3] and Polo and Tilouine [PT] that the formal characters of these cohomology groups are the same as in characteristic zero. We will see in Section 4 that for $H^2(\mathbf{u}, k)$ this is also true for $p > 3$ (and in some cases when $p = 3$). Let $w \in W$ have length 2. Then $w = s_\alpha s_\beta$ for some $\alpha, \beta \in \Delta$. Notice that

$$-w \cdot 0 = -(s_\alpha s_\beta) \cdot 0 = (1 - \langle \beta, \alpha^\vee \rangle) \alpha + \beta.$$

Furthermore, $\beta - \langle \beta, \alpha^\vee \rangle \alpha$ is a (positive) root. One can now conclude the following.

Corollary *Let $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$. Then $\phi_\alpha \wedge \phi_{-\langle \beta, \alpha^\vee \rangle \alpha + \beta}$ has weight $-w \cdot 0$ with $w = s_\alpha s_\beta$ and represents a cohomology class in $H^2(\mathbf{u}, k)$.*

2.4 We now identify some limitations on which other wedge products $\phi_\alpha \wedge \phi_\beta$ or linear combinations thereof can represent cohomology classes. Since the differentials are additive and preserve the action of T , of interest are linear combinations of wedge products that have the same weight. To avoid “trivial”

linear combinations, we say that an expression $\sum c_{\alpha,\beta}\phi_\alpha \wedge \phi_\beta \in \Lambda^2(\mathfrak{u}^*)$ is in *reduced form* if a pair (α, β) appears at most once and each $c_{\alpha,\beta} \neq 0$.

Proposition *Let $x = \sum c_{\alpha,\beta}\phi_\alpha \wedge \phi_\beta$ be an element in $\Lambda^2(\mathfrak{u}^*)$ in reduced form of weight γ for some $\gamma \in X(T)$. If $d_2(x) = 0$, then $d_1(\phi_\alpha) = 0$ for at least one α appearing in the sum.*

Proof. Observe that for any $\alpha \in \Phi^+$, if $d_1(\phi_\alpha) = \sum c_{\delta,\gamma}\phi_\delta \wedge \phi_\gamma$, then $\text{ht}(\delta) < \text{ht}(\alpha)$ and $\text{ht}(\gamma) < \text{ht}(\alpha)$ for all δ, γ . From all α and β appearing in the sum for x , choose a root σ with $\text{ht}(\sigma)$ being maximal. Without loss of generality, we may assume ϕ_σ appears in the second factor. Consider the corresponding term $c_{\alpha,\sigma}\phi_\alpha \wedge \phi_\sigma$. If we use the product rule to compute $d_2(x)$, one of the components will be $c_{\alpha,\sigma}d_1(\phi_\alpha) \wedge \phi_\sigma$. By height considerations, ϕ_σ appears in no other term. So this is not a linear combination of the other terms and we must have $d_1(\phi_\alpha) = 0$. \square

Combining this with Proposition 2.2, we immediately get the following.

Corollary *Assume that $p \geq 3$. For $p = 3$, assume further that Φ is not of type G_2 .*

- (a) *Let $x \in H^2(\mathfrak{u}, k)$ be a representative cohomology class in reduced form having weight γ for some $\gamma \in X(T)$. Then one of the components of x is of the form $\phi_\alpha \wedge \phi_\beta$ for some simple root $\alpha \in \Delta$ and positive root $\beta \in \Phi^+$ (with $\alpha + \beta = \gamma$).*
- (b) *Suppose $\phi_\alpha \wedge \phi_\beta$ represents a cohomology class in $H^2(\mathfrak{u}, k)$. Then α is a simple root and β is either a simple root or $\beta = \alpha + \sigma$ for some $\sigma \in \Phi^+$ and this is the unique decomposition of β into a sum of positive roots.*

Proof. Part (a) follows immediately from Propositions 2.2 and 2.4. For part (b), we again conclude, without loss of generality, that α must be simple. By assumption $d_2(\phi_\alpha \wedge \phi_\beta) = 0$. We have

$$d_2(\phi_\alpha \wedge \phi_\beta) = d_1(\phi_\alpha) \wedge \phi_\beta - \phi_\alpha \wedge d_1(\phi_\beta) = -\phi_\alpha \wedge d_1(\phi_\beta).$$

Hence, either $d_1(\phi_\beta) = 0$ or $d_1(\phi_\beta) = c\phi_\alpha \wedge \phi_\sigma$ for some $\sigma \in \Phi^+$ and non-zero constant c . In the first case, by Proposition 2.2, β would also have to be simple. The second case implies that $\beta = \alpha + \sigma$. Further, if $\beta = \sigma_1 + \sigma_2$ for another pair $\sigma_1, \sigma_2 \in \Phi^+$, then $d_1(\phi_\beta)$ would also contain $c'\phi_{\sigma_1} \wedge \phi_{\sigma_2}$ for some non-zero c' , contradicting $d_2(\phi_\alpha \wedge \phi_\beta) = 0$. \square

2.5 We make one final observation about weights that arise in $H^2(\mathfrak{u}, k)$.

Lemma *Assume that $p \geq 3$. If $p = 3$, assume further that Φ is not of type G_2 . Suppose $\gamma \in X(T)$ is a weight of $H^2(\mathfrak{u}, k)$ with $\gamma = i\beta_1 + \beta_2$ for $\beta_1, \beta_2 \in \Delta$ and $i > 0$. Then $i = 1 - \langle \beta_2, \beta_1^\vee \rangle$ and so $\gamma = -(s_{\beta_1} s_{\beta_2}) \cdot 0$.*

Proof. Since γ is a weight of $H^2(\mathfrak{u}, k)$ it must be a sum of positive roots. There is only one way this can happen: $\gamma = \beta_1 + ((i-1)\beta_1 + \beta_2)$. So we may assume that $(i-1)\beta_1 + \beta_2$ is a positive root. Since there is only one way in which γ can arise, it must be represented by the class $x_\gamma = \phi_{\beta_1} \wedge \phi_{(i-1)\beta_1 + \beta_2}$. From Proposition 2.2,

$$d_2(\phi_{\beta_1} \wedge \phi_{(i-1)\beta_1 + \beta_2}) = d_1(\phi_{\beta_1}) \wedge \phi_{(i-1)\beta_1 + \beta_2} - \phi_{\beta_1} \wedge d_1(\phi_{(i-2)\beta_1 + \beta_2}) = 0$$

since either $d_1(\phi_{(i-2)\beta_1 + \beta_2}) = 0$ or $d_1(\phi_{(i-2)\beta_1 + \beta_2}) = c\phi_{\beta_1} \wedge \phi_{(i-3)\beta_1 + \beta_2}$. So x_γ is indeed a cocycle.

We know that $(i-1)\beta_1 + \beta_2$ is a positive root. If $i\beta_1 + \beta_2$ is also a positive root, notice further that

$$d_1(\phi_{i\beta_1 + \beta_2}) = c'\phi_{\beta_1} \wedge \phi_{(i-1)\beta_1 + \beta_2} = c'x_\gamma$$

for a non-zero constant c' . Then x_γ would be a co-boundary and not represent a cohomology class. Hence, $i\beta_1 + \beta_2$ cannot be a positive root and we must have $i-1 = -\langle \beta_2, \beta_1^\vee \rangle$ as claimed. \square

3 Root Sums

3.1 As mentioned in the introduction, the computation of $H^2(\mathfrak{u}, k)$ involves a spectral sequence relating $H^2(\mathfrak{u}, k)$ and $H^2(U_1, k)$ (cf. Section 4.1) as well information about B_1 - and B -cohomology. In the process, certain equations involving sums of positive roots arise. This section is devoted to investigating these equations. The reader may wish to skip to Section 4 at this point to understand the general argument.

As above, we assume that $p \geq 3$. When $p = 3$ and Φ is of type G_2 , the general argument will not apply, and we omit that case from discussion here since the answers are not per se relevant to our discussion. Suppose $x \in H^2(\mathfrak{u}, k)$ has weight γ for some $\gamma \in X(T)$. From Corollary 2.4, we know that $\gamma = \alpha + \beta$ for some roots $\alpha \in \Delta$ and $\beta \in \Phi^+$, with $\alpha \neq \beta$. Given such roots α and β , we want to know whether or not there exists a weight $\sigma \in X(T)$, simple roots $\beta_1, \beta_2 \in \Delta$, and integers $0 \leq i \leq p-1$ and $m \geq 0$ such that any of the following hold:

$$\alpha + \beta = p\sigma \tag{3.1.1}$$

$$\alpha + \beta = \beta_1 + p\sigma \tag{3.1.2}$$

$$\alpha + \beta = i\beta_1 + p^m\beta_2 + p\sigma. \quad (3.1.3)$$

We briefly “explain” the relevance of these equations and refer the reader to Section 4 for more details. If γ is a weight of $H^2(\mathfrak{u}, k)$ and does not admit a solution to equations (3.1.1) and (3.1.2), then one can identify a weight ν with $H^2(B, -\gamma + p\nu) \neq 0$. Known information on B -cohomology, due to Andersen [And, 2.9] (cf. also [Jan1, II.12.5(a)]), then shows that γ must satisfy equation (3.1.3). Hence, knowing the possible solutions to (3.1.3) will place strong conditions on γ . Further, equation (3.1.1) also arises when considering the reduction $H^2(B_1, k) = H^2(U_1, k)^{T_1}$. And equation (3.1.2) will be directly relevant in understanding the vanishing of a certain differential in a spectral sequence (cf. Section 4.1).

In Propositions 3.1(A) and 3.1(B) we list all solutions to equations (3.1.1) and (3.1.2), respectively. We say that $\alpha + \beta$ is a trivial solution for (3.1.3) if the equation holds only for $m = 0$ and $\sigma = 0$. Notice that it follows from Lemma 2.5 that the trivial solutions of (3.1.3) give rise to the “classical” weights of the form $-w \cdot 0$ for $H^2(\mathfrak{u}, k)$. In Proposition 3.1(C) we list all non-trivial solutions $\alpha + \beta$ of (3.1.3) that are weights of $H^2(\mathfrak{u}, k)$. The propositions will be proved in Sections 3.2-3.5.

Proposition (A) *Let $p \geq 3$, $\alpha \in \Delta$, $\beta \in \Phi^+$, and $\alpha \neq \beta$. For $p = 3$, assume further that Φ is not of type G_2 .*

- (a) *Suppose $p > 3$. Then there is no weight $\sigma \in X(T)$ such that $\alpha + \beta = p\sigma$.*
- (b) *Suppose $p = 3$. The only cases in which there exists $\sigma \in X(T)$ with $\alpha + \beta = p\sigma$ are the following:*
 - (i) Φ is of type A_2 : $\alpha_1 + (\alpha_1 + \alpha_2) = 3\omega_1$.
 - (ii) Φ is of type A_2 : $\alpha_2 + (\alpha_1 + \alpha_2) = 3\omega_2$.

Proposition (B) *Let $p \geq 3$, $\alpha \in \Delta$, $\beta \in \Phi^+$, and $\alpha \neq \beta$. For $p = 3$, assume further that Φ is not of type G_2 .*

- (a) *Suppose $p > 3$. Then there is no simple root $\beta_1 \in \Delta$ and weight $\sigma \in X(T)$ such that $\alpha + \beta = \beta_1 + p\sigma$.*
- (b) *Suppose $p = 3$. The only cases in which there exists $\beta_1 \in \Delta$ and $\sigma \in X(T)$ with $\alpha + \beta = \beta_1 + p\sigma$ are the following:*
 - (i) Φ is of type B_n : $\alpha_n + (\alpha_{n-1} + 2\alpha_n) = \alpha_{n-1} + 3\alpha_n$.
 - (ii) Φ is of type C_n : $\alpha_{n-1} + (2\alpha_{n-1} + \alpha_n) = \alpha_n + 3\alpha_{n-1}$.
 - (iii) Φ is of type F_4 : $\alpha_3 + (\alpha_2 + 2\alpha_3) = \alpha_2 + 3\alpha_3$.

Proposition (C) *Let $p \geq 3$, $\alpha \in \Delta$, $\beta \in \Phi^+$, and $\alpha \neq \beta$. For $p = 3$, assume further that Φ is not of type G_2 .*

If $\alpha + \beta$ is a weight of $H^2(\mathfrak{u}, k)$ and there exist $\beta_1, \beta_2 \in \Delta$, $\sigma \in X(T)$, $0 < i < p$,

and $m \geq 0$ such that

$$\alpha + \beta = i\beta_1 + p^m\beta_2 + p\sigma,$$

then one of the following holds

- (a) $\alpha + \beta$ is a solution to equation (3.1.1) or (3.1.2),
- (b) $m = 0$ and $\sigma = 0$,

except in the following cases.

Roots α, β with $\alpha + \beta = -w \cdot 0$ for $w \in W$ with $l(w) = 2$:

- (i) $p = 5$, Φ is of type A_4 , and $\alpha + \beta = \alpha_1 + 2\alpha_2$ or $\alpha + \beta = 2\alpha_3 + \alpha_4$.
- (ii) $p = 3$, Φ is of type A_5 , and $\alpha + \beta = 2\alpha_1 + \alpha_2$ or $\alpha + \beta = \alpha_4 + 2\alpha_5$.
- (iii) $p = 3$, Φ is of type A_5 , and $\alpha + \beta = \alpha_1 + 2\alpha_2$ or $\alpha + \beta = 2\alpha_4 + \alpha_5$.
- (iv) $p = 3$, Φ is of type A_5 , and $\alpha + \beta = \alpha_1 + \alpha_4$ or $\alpha + \beta = \alpha_2 + \alpha_5$.
- (v) $p = 3$, Φ is of type E_6 , and $\alpha + \beta = 2\alpha_1 + \alpha_3$ or $\alpha + \beta = \alpha_5 + 2\alpha_6$.
- (vi) $p = 3$, Φ is of type E_6 , and $\alpha + \beta = \alpha_1 + 2\alpha_3$ or $\alpha + \beta = 2\alpha_5 + \alpha_6$.
- (vii) $p = 3$, Φ is of type E_6 , and $\alpha + \beta = \alpha_1 + \alpha_5$ or $\alpha + \beta = \alpha_3 + \alpha_6$.

Roots α, β with $\alpha + \beta \neq -w \cdot 0$ for $w \in W$ with $l(w) = 2$:

- (viii) $p = 3$, Φ is of type B_n , $n \geq 3$, and $\alpha + \beta = \alpha_{n-2} + 2\alpha_{n-1} + 3\alpha_n$ corresponding to the cohomology class

$$\phi_{\alpha_n} \wedge \phi_{\alpha_{n-2} + 2\alpha_{n-1} + 2\alpha_n} - \phi_{\alpha_{n-1} + \alpha_n} \wedge \phi_{\alpha_{n-2} + \alpha_{n-1} + 2\alpha_n} + \phi_{\alpha_{n-2} + \alpha_{n-1} + \alpha_n} \wedge \phi_{\alpha_{n-1} + 2\alpha_n}.$$

- (ix) $p = 3$, Φ is of type C_n , $n \geq 3$, and $\alpha + \beta = \alpha_{n-2} + 3\alpha_{n-1} + \alpha_n$ corresponding to the cohomology class

$$\phi_{\alpha_{n-1}} \wedge \phi_{\alpha_{n-2} + 2\alpha_{n-1} + \alpha_n} - \phi_{\alpha_{n-2} + \alpha_{n-1}} \wedge \phi_{2\alpha_{n-1} + \alpha_n}.$$

- (x) $p = 3$, Φ is of type F_4 , and $\alpha + \beta = \alpha_1 + 2\alpha_2 + 3\alpha_3$ corresponding to

$$\phi_{\alpha_3} \wedge \phi_{\alpha_1 + 2\alpha_2 + 2\alpha_3} - \phi_{\alpha_2 + \alpha_3} \wedge \phi_{\alpha_1 + \alpha_2 + 2\alpha_3} + \phi_{\alpha_1 + \alpha_2 + \alpha_3} \wedge \phi_{\alpha_2 + 2\alpha_3}$$

or $\alpha + \beta = \alpha_2 + 3\alpha_3 + \alpha_4$ corresponding to the cohomology class

$$\phi_{\alpha_3} \wedge \phi_{\alpha_2 + 2\alpha_3 + \alpha_4} - \phi_{\alpha_3 + \alpha_4} \wedge \phi_{\alpha_2 + 2\alpha_3}.$$

Remark In parts (i)-(vii) of Proposition (C), the solutions $\alpha + \beta$ arise from distinct pairs $w, w' \in W$ with $l(w) = 2 = l(w')$ and $-w \cdot 0 = -w' \cdot 0 + p\sigma$ for some $\sigma \in X(T)$. Each part contains a corresponding pair.

3.2 We now proceed to prove the propositions in Section 3.1. Assume $p \geq 3$ and let $\alpha \in \Delta$ and $\beta \in \Phi^+$. We first look at the situation where σ is contained in the root lattice. We can guarantee that this occurs if $p \nmid (X(T) : \mathbb{Z}\Phi)$. For

$p \geq 3$ this happens except when $\Phi = A_n$ and $p \mid n + 1$ or $\Phi = E_6$ and $p = 3$. Now express $\sigma = \sum_{i=1}^n m_i \alpha_i$. It is clear that (3.1.1)-(3.1.3) will have a trivial solution if p is greater than or equal to N where N is two plus the largest coefficient appearing in the expression of a positive root in terms of simple roots. For each root system, we have listed N below.

Φ	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
N	3	4	4	4	5	6	8	6	5

For $\alpha \in \Delta$ and $\beta \in \Phi^+$, express $\alpha + \beta = \sum_{i=1}^n l_i \alpha_i$. If we have a non-trivial solution of (3.1.1)-(3.1.3), then at least $n - 2$ of the elements in $\{l_i \mid i = 1, 2, \dots, n\}$ must be congruent to zero modulo p . It can be easily verified that this cannot happen in the cases when Φ is of type D_n ($p = 3$), E_6 ($p = 3$), E_7 ($p = 3, 5$), E_8 ($p = 3, 5, 7$), F_4 ($p = 5$). We are now reduced to looking at the cases when $\Phi = B_n, C_n, F_4$ for $p = 3$ and finishing off our analysis when $\Phi = A_n$ and $p \mid n + 1$, or $\Phi = E_6$ and $p = 3$.

3.3 Types B_n, C_n, F_4 : Let us look at the cases when $p = 3$ and $\Phi = B_n, C_n$, or F_4 . First consider Proposition 3.1(A) corresponding to equation (3.1.1). Write $\beta = \sum_{i=1}^n n_i \alpha_i$. To have a solution, at most one of the n_i can be non-zero mod 3. The only such positive roots β satisfying that condition are simple roots. But it is straightforward to check that a sum of simples is never equal to 3σ .

Next consider Proposition 3.1(B) corresponding to equation (3.1.2). Arguing as above, in this case, at most two of the n_i can be non-zero mod 3. For type B_n , the possible such β are the α_i for $1 \leq i \leq n$, or $\alpha_i + \alpha_{i+1}$ for $1 \leq i \leq n - 1$, or $\alpha_{n-1} + 2\alpha_n$. Direct verification shows that the first two types do not yield a solution but the last one does. Types C_n and F_4 are analogous with the “special” roots instead being $2\alpha_{n-1} + \alpha_n$ and $\alpha_2 + 2\alpha_3$ respectively.

Finally, consider Proposition 3.1(C) corresponding to equation (3.1.3) The case $\Phi = B_2 = C_2$ can be handled by direct computation. Consider $\Phi = B_n$ for $n \geq 3$. The above solution to (3.1.2) is also a solution to (3.1.3) as listed in part (b) of the proposition. The only other positive roots α, β for which $\alpha + \beta$ can have a non-trivial solution to (3.1.3) are

$$\begin{aligned} \alpha &= \alpha_n, & \beta &= \gamma_1 = \alpha_{n-2} + \alpha_{n-2} + 2\alpha_n \\ \alpha &= \alpha_{n-1} \text{ or } \alpha_n, & \beta &= \gamma_2 = \alpha_{n-2} + 2\alpha_{n-1} + 2\alpha_n. \end{aligned}$$

For the weights $\alpha_n + \gamma_1$ and $\alpha_{n-1} + \gamma_2$, the weight space in $\Lambda^2(\mathfrak{u}^*)$ is only one dimensional - spanned by $\phi_{\alpha_n} \wedge \phi_{\gamma_1}$ and $\phi_{\alpha_{n-1}} \wedge \phi_{\gamma_2}$ respectively. Since γ_1 and γ_2 can be expressed as a sum of positive roots in more than one way, by Corollary

2.4(b), these cannot be cocycles. The last weight, $\alpha_n + \gamma_2 = \alpha_{n-2} + 2\alpha_{n-1} + 3\alpha_n$, does yield a new class in $H^2(\mathfrak{u}, k)$:

$$\phi_{\alpha_n} \wedge \phi_{\alpha_{n-2} + 2\alpha_{n-1} + 2\alpha_n} - \phi_{\alpha_{n-1} + \alpha_n} \wedge \phi_{\alpha_{n-2} + \alpha_{n-1} + 2\alpha_n} + \phi_{\alpha_{n-2} + \alpha_{n-1} + \alpha_n} \wedge \phi_{\alpha_{n-1} + 2\alpha_n}.$$

This was confirmed by using the computer package MAGMA [BC],[BCP] to construct a Chevalley basis for B_3 and direct computation of the differentials.

Let $\Phi = C_n$ where $n \geq 3$. This case is similar to the situation for B_n . Again, we get the solution to (3.1.2) which is also a solution to (3.1.3). The only weights $\alpha + \beta$ which can have a non-trivial solution to (3.1.3) are

$$\begin{aligned} \alpha = \alpha_{n-2} \text{ or } \alpha_{n-1}, & \quad \beta = \gamma_1 = 2\alpha_{n-2} + 2\alpha_{n-1} + \alpha_n \\ \alpha = \alpha_{n-1}, & \quad \beta = \gamma_2 = \alpha_{n-2} + 2\alpha_{n-1} + \alpha_n. \end{aligned}$$

The first weight $\alpha_{n-2} + \gamma_1$ has a one dimensional weight space and γ_1 can be expressed as a sum of positive roots in more than one way. So it does not yield a new cohomology class. The second weight $\alpha_{n-1} + \gamma_1$ has a two dimensional weight space represented by $\phi_{\alpha_{n-1}} \wedge \phi_{\gamma_1}$ and $\phi_{\alpha_{n-2} + \alpha_{n-1}} \wedge \phi_{\alpha_{n-2} + 2\alpha_{n-1} + \alpha_{n-1}}$, but no non-trivial linear combination of those two gives a cocycle. (In the Lie algebra, $[x_{\alpha_{n-2}}, x_{2\alpha_{n-1} + \alpha_{n-1}}] = \pm x_{\alpha_{n-2} + 2\alpha_{n-1} + \alpha_{n-1}}$, but there is no positive root σ with $[x_\sigma, x_{2\alpha_{n-1} + \alpha_{n-1}}] = cx_{\gamma_1}$ for a non-zero c .) The last weight $\alpha_{n-1} + \gamma_2 = \alpha_{n-2} + 3\alpha_{n-1} + \alpha_n$ does yield a new class in $H^2(\mathfrak{u}, k)$:

$$\phi_{\alpha_{n-1}} \wedge \phi_{\alpha_{n-2} + 2\alpha_{n-1} + \alpha_n} - \phi_{\alpha_{n-2} + \alpha_{n-1}} \wedge \phi_{2\alpha_{n-1} + \alpha_n} \quad (3.3.1)$$

This is again confirmed by constructing a Chevalley basis using MAGMA.

Let $\Phi = F_4$. Once again we need to look at possible solutions $\alpha + \beta$ of (3.1.3) which are not already solutions to (3.1.2). This possibly occurs for

$$\begin{aligned} \alpha = \alpha_1 \text{ or } \alpha_4, & \quad \beta = \gamma_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 \\ \alpha = \alpha_4, & \quad \beta = \gamma_2 = \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 \\ \alpha = \alpha_2 \text{ or } \alpha_4, & \quad \beta = \gamma_3 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 \\ \alpha = \alpha_2 & \quad \beta = \gamma_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4 \\ \alpha = \alpha_2 \text{ or } \alpha_3, & \quad \beta = \gamma_5 = \alpha_1 + 2\alpha_2 + 2\alpha_3 \\ \alpha = \alpha_3, & \quad \beta = \gamma_6 = \alpha_1 + \alpha_2 + 2\alpha_3 \\ \alpha = \alpha_3 \text{ or } \alpha_4, & \quad \beta = \gamma_7 = \alpha_2 + 2\alpha_3 + 2\alpha_4 \\ \alpha = \alpha_3, & \quad \beta = \gamma_8 = \alpha_2 + 2\alpha_3 + \alpha_4. \end{aligned}$$

For γ_j , $j = 1, 2, 3, 4$, one can use arguments as above along with Corollary 2.4 to see that if $d_2(x) = 0$ for a general cohomology class $x = \sum_{\sigma_1 + \sigma_2 = \sigma} c_{\sigma_1, \sigma_2} \phi_{\sigma_1} \wedge \phi_{\sigma_2}$ with $\sigma = \alpha + \beta$, then $c_{\sigma_1, \sigma_2} = 0$ for all pairs (σ_1, σ_2) with $\sigma_1 + \sigma_2 = \sigma$. In the case of $\alpha_4 + \gamma_4$, one must actually use a Chevalley basis constructed by MAGMA [BC,BCP] and explicit computation of d_2 . For γ_j , $j = 5, 6, 7, 8$, the

possibilities for cohomology classes in F_4 arise from cohomology classes in the Levi factors of types B_3 or C_3 . From the results above we can conclude that

$$\phi_{\alpha_3} \wedge \phi_{\alpha_1+2\alpha_2+2\alpha_3} - \phi_{\alpha_2+\alpha_3} \wedge \phi_{\alpha_1+\alpha_2+2\alpha_3} + \phi_{\alpha_1+\alpha_2+\alpha_3} \wedge \phi_{\alpha_2+2\alpha_3}, \quad (3.3.2)$$

$$\phi_{\alpha_3} \wedge \phi_{\alpha_2+2\alpha_3+\alpha_4} - \phi_{\alpha_3+\alpha_4} \wedge \phi_{\alpha_2+2\alpha_3} \quad (3.3.3)$$

are new classes in F_4 of weights $\alpha_1 + 2\alpha_2 + 3\alpha_3$ and $\alpha_2 + 3\alpha_3 + \alpha_4$.

3.4 Type A_n : Let us look at the situation where $p \mid n + 1$ with $\Phi = A_n$. Then $X(T)/\mathbb{Z}\Phi \cong \mathbb{Z}_{n+1}$. Moreover, $X(T)/\mathbb{Z}\Phi = \{t\omega_1 + \mathbb{Z}\Phi : t = 0, 1, \dots, n\}$. Now

$$t\omega_1 = \frac{t}{n+1}(n\alpha_1 + (n-1)\alpha_2 + \dots + \alpha_n).$$

One can revise (3.1.1)-(3.1.3) in the following way. We are looking for $\alpha \in \Delta$, $\beta \in \Phi^+$ satisfying

$$\alpha + \beta = pt\omega_1 + p\sigma \quad (3.4.1)$$

$$\alpha + \beta = \beta_1 + pt\omega_1 + p\sigma \quad (3.4.2)$$

$$\alpha + \beta = i\beta_1 + p^m\beta_2 + pt\omega_1 + p\sigma, \quad (3.4.3)$$

where $\sigma \in \mathbb{Z}\Phi$. Since $pt\omega_1$ must lie in $\mathbb{Z}\Phi$, $\frac{pt}{n+1} \in \mathbb{Z}$. Since $p \mid n + 1$, it follows that $\frac{t}{s} \in \mathbb{Z}$ where $s := \frac{n+1}{p}$. If $p \mid \frac{t}{s}$ then we are done because (3.4.1)-(3.4.3) would reduce to the original (3.1.1)-(3.1.3) with σ lying in the root lattice. And the arguments in Section 3.2 would apply. Therefore, we may assume that $\frac{t}{s}$ is not congruent to zero modulo p . Consider $\alpha + \beta = \sum_{i=1}^n m_i \alpha_i$. Then $m_i \in \{0, 1, 2\}$ for $i = 1, 2, \dots, n$ and m_i can be 2 for at most one i . For $p \geq 7$, (3.4.1)-(3.4.3) must have a trivial solution because on the right hand side we must have a number which is not congruent to $\{0, 1, 2\}$ modulo p . For $p = 5$, if $n + 1 \geq 10$ the same must happen, so we are reduced to understanding $p = 5$, $\Phi = A_4$ and the cases when $p = 3$. When $p = 3$ we can eliminate the cases when $n + 1 \geq 9$ by using congruences. This reduces us to looking at A_5 and A_2 when $p = 3$.

Consider first the case of A_4 when $p = 5$. For (3.1.1), it is easy to see that $\alpha + \beta \neq 5\sigma$ since when α and β are expressed as sums of fundamental weights, the coefficients are $-1, 0, 1$, or 2 . For (3.1.2), suppose $\alpha + \beta = \beta_1 + 5\sigma$. Note that β_1 cannot be α . Write $\beta_1 = \sum_{i=1}^4 c_i \omega_i$. Note that $c_j = 2$ for some j . Hence the coefficients of ω_j in α and β must sum to $2 \pmod{5}$. Since α is simple it has coefficients of $-1, 0$, and 2 . Hence the only way to add to $2 \pmod{5}$ is if $\alpha = \alpha_j = \beta_1$ which has already been ruled out. For (3.1.3), direct calculations of $H^2(\mathfrak{u}, k)$ show that all weights have the form $-w \cdot 0$ for $w \in W$ with $l(w) = 2$. As noted in Section 2.3, such weights are of the form $i\beta_1 + \beta_2$. So the only way to get a non-trivial solution to (3.1.3) is if $-w \cdot 0 = -w' \cdot 0 \pmod{pX(T)}$ for distinct pairs $w, w' \in W$ with $l(w) = 2$. Checking all cases, one finds one solution: $-(s_{\alpha_2} s_{\alpha_1}) \cdot 0 = \alpha_1 + 2\alpha_2 = 2\alpha_3 + \alpha_4 + 5(\omega_2 - \omega_3) = -(s_{\alpha_3} s_{\alpha_4}) \cdot 0 + 5(\omega_2 - \omega_3)$.

Next consider the case of A_5 when $p = 3$. It is again straightforward to check that there are no solutions to (3.1.1). For (3.1.2), expressing α , β , and β_1 in terms of fundamental weights, one can directly check that there are no solutions. Again for (3.1.3), direct calculation of $H^2(\mathfrak{u}, k)$ shows that all weights have the form $-w \cdot 0$ for $w \in W$ with $l(w) = 2$. There are three cases where $-w \cdot 0 = -w' \cdot 0 \pmod{pX(T)}$:

$$\begin{aligned} -(s_{\alpha_1} s_{\alpha_2}) \cdot 0 &= 2\alpha_1 + \alpha_2 = \alpha_4 + 2\alpha_5 + 3(\omega_1 - \omega_5) \\ &= -(s_{\alpha_5} s_{\alpha_4}) \cdot 0 + 3(\omega_1 - \omega_5) \\ -(s_{\alpha_2} s_{\alpha_1}) \cdot 0 &= \alpha_1 + 2\alpha_2 = 2\alpha_4 + \alpha_5 + 3(\omega_2 - \omega_4) \\ &= -(s_{\alpha_4} s_{\alpha_5}) \cdot 0 + 3(\omega_2 - \omega_4) \\ -(s_{\alpha_1} s_{\alpha_4}) \cdot 0 &= \alpha_1 + \alpha_4 = \alpha_2 + \alpha_5 + 3(\omega_1 - \omega_2 + \omega_4 - \omega_5) \\ &= -(s_{\alpha_2} s_{\alpha_5}) \cdot 0 + 3(\omega_1 - \omega_2 + \omega_4 - \omega_5). \end{aligned}$$

Lastly, the case of A_2 is easily verified by hand. Note that the solutions to (3.1.1) are also solutions to (3.1.3).

3.5 Type E_6 : Consider the case when $\Phi = E_6$ and $p = 3$. We have $X(T)/\mathbb{Z}\Phi \cong \mathbb{Z}_3$ and $X(T)/\mathbb{Z}\Phi = \{\mathbb{Z}\Phi, \omega_1 + \mathbb{Z}\Phi, 2\omega_1 + \mathbb{Z}\Phi\}$. In terms of the root basis one has

$$\omega_1 = \frac{1}{3}(4\alpha_1 + 3\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6).$$

This forces us to revise (3.1.1)-(3.1.3) in the following way. We are looking for $\alpha \in \Delta$, $\beta \in \Phi^+$ satisfying

$$\alpha + \beta = pt\omega_1 + p\sigma \tag{3.5.1}$$

$$\alpha + \beta = \beta_1 + pt\omega_1 + p\sigma \tag{3.5.2}$$

$$\alpha + \beta = i\beta_1 + p^m\beta_2 + pt\omega_1 + p\sigma, \tag{3.5.3}$$

where $\sigma \in \mathbb{Z}\Phi$ and $t = 0, 1, 2$. The cases when $t = 0$ reduce to (3.1.1)-(3.1.3) with σ lying in the root lattice. The arguments in Section 3.2 can be applied to show that (3.5.1)-(3.5.3) have a trivial solution. For $t = 1$, one can see that (3.5.1) and (3.5.2) have to have trivial solutions. So we need to look at possible solutions $\alpha + \beta$ of (3.5.3). If $\alpha + \beta \neq -w \cdot 0$ for $w \in W$ with $l(w) = 2$, then

the only possible values are:

$$\begin{array}{ll}
\alpha = \alpha_2 \text{ or } \alpha_6, & \beta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 \\
\alpha = \alpha_6, & \beta = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 \\
\alpha = \alpha_4 \text{ or } \alpha_6, & \beta = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 \\
\alpha = \alpha_4, & \beta = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 \\
\alpha = \alpha_3 \text{ or } \alpha_6, & \beta = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 \\
\alpha = \alpha_3, & \beta = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 \\
\alpha = \alpha_1, & \beta = \alpha_5 + \alpha_6 \\
\alpha = \alpha_5, & \beta = \alpha_1 + \alpha_3.
\end{array}$$

In each case, one can directly verify using Corollary 2.4 that there are no cohomology classes of these weights. The $t = 2$ case is handled using the same procedure.

As for type A_n , there are three sets of pairs $w, w' \in W$ with $l(w) = 2 = l(w')$ and $-w \cdot 0 = -w' \cdot 0 + p\sigma$ for some $\sigma \in X(T)$. They are as follows:

$$\begin{aligned}
-(s_{\alpha_1} s_{\alpha_3}) \cdot 0 &= 2\alpha_1 + \alpha_3 = \alpha_5 + 2\alpha_6 + 3(\omega_1 - \omega_6) \\
&= -(s_{\alpha_6} s_{\alpha_5}) \cdot 0 + 3(\omega_1 - \omega_6) \\
-(s_{\alpha_3} s_{\alpha_1}) \cdot 0 &= \alpha_1 + 2\alpha_3 = 2\alpha_5 + \alpha_6 + 3(\omega_3 - \omega_5) \\
&= -(s_{\alpha_5} s_{\alpha_6}) \cdot 0 + 3(\omega_3 - \omega_5) \\
-(s_{\alpha_1} s_{\alpha_5}) \cdot 0 &= \alpha_1 + \alpha_5 = \alpha_3 + \alpha_6 + 3(\omega_1 - \omega_3 + \omega_5 - \omega_6) \\
&= -(s_{\alpha_3} s_{\alpha_6}) \cdot 0 + 3(\omega_1 - \omega_3 + \omega_5 - \omega_6).
\end{aligned}$$

4 Lie algebra cohomology

We recall from the introduction that the series of reduction steps to compute $H^2(G_r, H^0(\lambda))$ has as its foundation the computation of $H^2(\mathfrak{u}, k)$. The goal of this section is to identify $H^2(\mathfrak{u}, k)$ for $p \geq 3$ (see Theorem 4.4). This will be done using the root sum computations of Section 3 and information about B_1 - and B -cohomology.

4.1 Relating \mathfrak{u} and U_1 : We begin by relating $H^2(\mathfrak{u}, k)$ to $H^2(U_1, k)$ via the first quadrant spectral sequence introduced by Friedlander and Parshall [FP1] for $p \geq 3$ (and later generalized by Andersen and Jantzen [AJ] and Friedlander and Parshall [FP3]; cf. also [Jan1, I.9.20]):

$$E_2^{2i,j} = S^i(\mathfrak{u}^*)^{(1)} \otimes H^j(\mathfrak{u}, k) \Rightarrow H^{2i+j}(U_1, k). \quad (4.1.1)$$

The maximal torus T acts on the spectral sequence and all differentials are T -homomorphisms. Since the odd columns are all zero, the only terms that

can contribute to $H^2(U_1, k)$ are $E_2^{2,0} = (\mathbf{u}^*)^{(1)}$ and $E_2^{0,2} = H^2(\mathbf{u}, k)$. Since the T -action is preserved, by Proposition 2.2, the differential $d_2 : E_2^{0,1} = H^1(\mathbf{u}, k) \rightarrow (\mathbf{u}^*)^{(1)} = E_2^{2,0}$ must be zero. Hence the $E_2^{2,0}$ -term consists of universal cycles and so, as B -modules,

$$H^2(U_1, k) \supset (\mathbf{u}^*)^{(1)}. \quad (4.1.2)$$

Of key importance will be the differential $d_2 : H^2(\mathbf{u}, k) = E_2^{0,2} \rightarrow E_2^{1,1} = H^1(\mathbf{u}, k) \otimes (\mathbf{u}^*)^{(1)}$. The nature of the weights involved gives rise to equation (3.1.2).

To investigate this, we observe that the spectral sequence can be refined. Since weight spaces are preserved by the differentials, and all modules for T are completely reducible, for each $\lambda \in X(T)$, one obtains a spectral sequence:

$$E_2^{2i,j} = [S^i(\mathbf{u}^*)^{(1)} \otimes H^j(\mathbf{u}, k)]_\lambda \Rightarrow H^{2i+j}(U_1, k)_\lambda. \quad (4.1.3)$$

Using this spectral sequence one gets the following result.

Proposition *Let $p \geq 3$ and $\lambda \in X(T)$. As a T -module,*

$$H^2(U_1, k)_\lambda \cong H^2(\mathbf{u}, k)_\lambda \oplus ((\mathbf{u}^*)^{(1)})_\lambda \quad (4.1.4)$$

except for the following weights

- (i) $p = 3$, Φ is of type B_n : $\lambda = \alpha_{n-1} + 3\alpha_n = -(s_{\alpha_n} s_{\alpha_{n-1}}) \cdot 0$,
- (ii) $p = 3$, Φ is of type C_n : $\lambda = 3\alpha_{n-1} + \alpha_n = -(s_{\alpha_{n-1}} s_{\alpha_n}) \cdot 0$,
- (iii) $p = 3$, Φ is of type F_4 : $\lambda = \alpha_2 + 3\alpha_3 = -(s_{\alpha_3} s_{\alpha_2}) \cdot 0$,
- (iv) $p = 3$, Φ is of type G_2 : $\lambda = 3\alpha_1 + \alpha_2$,

where $H^2(U_1, k)_\lambda$ is a T -submodule of $H^2(\mathbf{u}, k)_\lambda$.

Proof. Consider the spectral sequence (4.1.3). The only terms that can contribute to $H^2(U_1, k)_\lambda$ are $E_2^{2,0} = ((\mathbf{u}^*)^{(1)})_\lambda$ and $E_2^{0,2} = H^2(\mathbf{u}, k)_\lambda$. From (4.1.2) one obtains

$$H^2(U_1, k)_\lambda \supset ((\mathbf{u}^*)^{(1)})_\lambda.$$

Whether or not the term $E_2^{0,2}$ contributes depends upon the differential

$$d_2 : H^2(\mathbf{u}, k)_\lambda = E_2^{0,2} \rightarrow E_2^{1,1} = ((\mathbf{u}^*)^{(1)} \otimes H^1(\mathbf{u}, k))_\lambda.$$

Suppose first that $p \neq 3$ or Φ is not of type G_2 . It follows from Corollary 2.4 that a weight of $H^2(\mathbf{u}, k)$ must be of the form $\alpha + \beta$ for $\alpha \in \Delta$ and $\beta \in \Phi^+$. On the other hand, from Proposition 2.2, a weight of $(\mathbf{u}^*)^{(1)} \otimes H^1(\mathbf{u}, k)$ must be of the form $p\sigma + \beta_1$ for $\sigma \in \Phi^+$ and $\beta_1 \in \Delta$. To have a non-trivial differential, we would need $\alpha + \beta = p\sigma + \beta_1$ which is precisely equation (3.1.2). From Proposition 3.1(B), there are no solutions except for the excluded weights λ in parts (i)-(iii). Hence the differential must always be zero and (4.1.4)

holds. Note that for an excluded weight λ , $E_2^{2,0} = ((\mathbf{u}^*)^{(1)})_\lambda = 0$ and so $H^2(U_1, k)_\lambda \subset E_2^{0,2} = H^2(\mathbf{u}, k)_\lambda$ as claimed.

If $p = 3$ and Φ is of type G_2 , a straightforward calculation shows that the only weight of $H^2(\mathbf{u}, k)$ that could possibly result in a non-zero map d_2 is $3\alpha_1 + \alpha_2$. In this case, a complete list of T -module basis elements for $H^2(\mathbf{u}, k)$ can be found in Section 4.4. \square

4.2 Relating \mathbf{u} and B_1 : Using the Lyndon-Hochschild-Serre (LHS) spectral sequence, cohomology for U_1 can be readily related to that for B_1 . Combining this with the observation in the previous section, one can relate \mathbf{u} -cohomology with B_1 -cohomology. The following proposition will allow us to completely determine $H^2(\mathbf{u}, k)$ in Section 4.4, which will also provide information to compute $H^2(B_1, \lambda)$ for all $\lambda \in X(T)$. This will be done in Section 5.

Proposition *Let $p \geq 3$ and $\lambda \in X(T)$, such that $\lambda \notin pX(T)$. As a T -module,*

$$H^2(B_1, \lambda) \cong \bigoplus_{\nu \in X(T)} p\nu^{\dim H^2(\mathbf{u}, k) - \lambda + p\nu}$$

except for the following cases

- (i) $p = 3$, $\Phi = B_n$: $\lambda \equiv (s_{\alpha_n} s_{\alpha_{n-1}}) \cdot 0 \pmod{pX(T)}$,
- (ii) $p = 3$, $\Phi = C_n$: $\lambda \equiv (s_{\alpha_{n-1}} s_{\alpha_n}) \cdot 0 \pmod{pX(T)}$,
- (iii) $p = 3$, $\Phi = F_4$: $\lambda \equiv (s_{\alpha_3} s_{\alpha_2}) \cdot 0 \pmod{pX(T)}$,
- (iv) $p = 3$, $\Phi = G_2$: $\lambda \equiv -(3\alpha_1 + \alpha_2) \pmod{pX(T)}$,

where

$$H^2(B_1, \lambda) \subset \bigoplus_{\nu \in X(T)} p\nu^{\dim H^2(\mathbf{u}, k) - \lambda + p\nu}.$$

Proof. The B_1 -cohomology can easily be related to U_1 -cohomology. Using the LHS spectral sequence for $U_1 \trianglelefteq B_1$ and the fact that modules over $T_1 = B_1/U_1$ are completely reducible, one concludes that for $i \geq 0$

$$H^i(B_1, \lambda) \cong H^i(U_1, \lambda)^{T_1} \cong \left(H^i(U_1, k) \otimes \lambda \right)^{T_1}. \quad (4.2.1)$$

It suffices to determine the $-\lambda$ weight space of $H^2(U_1, k)$ relative to T_1 . The T_1 -action on $H^\bullet(U_1, k)$ extends to T so the aforementioned condition is tantamount to determining the T -weights μ of $H^2(U_1, k)$ of the form $\mu = -\lambda + p\nu$ for some $\nu \in X(T)$. We have

$$\begin{aligned} \mathrm{H}^2(B_1, \lambda) &\cong \left(\mathrm{H}^2(U_1, k) \otimes \lambda \right)^{T_1} \cong \bigoplus_{\nu \in X(T)} \left(\mathrm{H}^2(U_1, k) \otimes (\lambda - p\nu) \right)^T \otimes p\nu \\ &\cong \bigoplus_{\nu \in X(T)} p\nu^{\dim \mathrm{H}^2(U_1, k) - \lambda + p\nu}. \end{aligned}$$

The assertion follows from Proposition 4.1 and the fact that $\lambda \notin pX(T)$. \square

4.3 Relating \mathfrak{u} and B : With the aid of the previous proposition and information about B -cohomology, we can now show that almost all weights of $\mathrm{H}^2(\mathfrak{u}, k)$ have the form $-w \cdot 0$ for $w \in W$ with $l(w) = 2$.

Proposition *Let $p \geq 3$ and $\gamma \in X(T)$ be a weight of $\mathrm{H}^2(\mathfrak{u}, k)$. For $p = 3$, assume further that Φ is not of type G_2 . Then $\gamma = -w \cdot 0$ for some $w \in W$ with $l(w) = 2$, except in the following cases:*

- (i) $p = 3$ and Φ is of type B_n , $n \geq 3$: $\gamma = \alpha_{n-2} + 2\alpha_{n-1} + 3\alpha_n$
- (ii) $p = 3$ and Φ of type C_n , $n \geq 3$: $\gamma = \alpha_{n-2} + 3\alpha_{n-1} + \alpha_n$
- (iii) $p = 3$ and Φ of type F_4 : $\gamma = \alpha_1 + 2\alpha_2 + 3\alpha_3$ or $\gamma = \alpha_2 + 3\alpha_3 + \alpha_4$.

Proof. By Corollary 2.4 any weight γ of $\mathrm{H}^2(\mathfrak{u}, k)$ is of the form $\alpha + \beta$ with $\alpha \in \Delta$, $\beta \in \Phi^+$, and $\alpha \neq \beta$. We consider three cases.

Case 1: Assume that $\gamma = \alpha + \beta \in pX(T)$, i.e. γ is a solution to equation (3.1.1).

From Proposition 3.1(A) we conclude that

- (i) $p = 3$, Φ is of type A_2 and $\gamma = \alpha_1 + (\alpha_1 + \alpha_2) = -(s_{\alpha_1} s_{\alpha_2}) \cdot 0$
- (ii) $p = 3$, Φ is of type A_2 and $\gamma = \alpha_2 + (\alpha_1 + \alpha_2) = -(s_{\alpha_2} s_{\alpha_1}) \cdot 0$.

Case 2: Assume that $\gamma = \alpha + \beta$ is a solution to equation (3.1.2).

Here Proposition 3.1 (B) implies that

- (i) $p = 3$, Φ is of type B_n , $n \geq 3$, and $\gamma = \alpha_n + (\alpha_{n-1} + 2\alpha_n) = -(s_{\alpha_n} s_{\alpha_{n-1}}) \cdot 0$
- (ii) $p = 3$, Φ is of type C_n , $n \geq 3$, and $\gamma = \alpha_{n-1} + (2\alpha_{n-1} + \alpha_n) = -(s_{\alpha_{n-1}} s_{\alpha_n}) \cdot 0$
- (iii) $p = 3$, Φ is of type F_4 , and $\gamma = \alpha_3 + (\alpha_2 + 2\alpha_3) = -(s_{\alpha_3} s_{\alpha_2}) \cdot 0$.

Case 3: Assume that $\gamma = \alpha + \beta$ is neither a solution to (3.1.1) nor to (3.1.2). First we apply Proposition 4.2 and conclude that $\mathrm{H}^2(B_1, -\gamma) \neq 0$. Next we consider the LHS spectral sequence applied to $B_1 \trianglelefteq B$ with $\nu \in X(T)$:

$$E_2^{i,j} = \mathrm{Ext}_{B/B_1}^i(-p\nu, \mathrm{H}^j(B_1, -\gamma)) \Rightarrow \mathrm{H}^{i+j}(B, -\gamma + p\nu).$$

Since $\gamma \notin pX(T)$, we have $E_2^{i,0} = 0$ for $i \geq 0$ so in particular $E_2^{2,0} = 0$. If $E_2^{i,1} \neq 0$ then $\mathrm{H}^1(B_1, -\gamma) \neq 0$ and by [Jan2, §3] we have $\gamma = \alpha + \beta = p\sigma + \beta_1$ for some $\sigma \in X(T)$ and $\beta_1 \in \Delta$. This implies that γ is a solution to equation

(3.1.2). Since we are excluding this case, we conclude that $E_2^{i,1} = 0$ for $i \geq 0$. It follows that for all $\nu \in X(T)$

$$\mathrm{H}^2(B, -\gamma + p\nu) \cong E_2^{0,2} = \mathrm{Hom}_{B/B_1}(-p\nu, \mathrm{H}^2(B_1, -\gamma)). \quad (4.3.1)$$

Since $\mathrm{H}^2(B_1, -\gamma) \neq 0$ there exists some $\sigma \in X(T)$ with

$$\mathrm{Hom}_{B/B_1}(-p\sigma, \mathrm{H}^2(B_1, -\gamma)) \neq 0,$$

or equivalently $\mathrm{H}^2(B, -\gamma + p\sigma) \neq 0$. The latter condition implies that $\gamma - p\sigma = i\beta_1 + p^m\beta_2$ for some $i > 0$, $m \geq 0$, and $\beta_1, \beta_2 \in \Delta$ by [And, 2.9] (cf. also [Jan1, II.12.5(a)]). This implies that $\gamma = \alpha + \beta$ is a solution to equation (3.1.3). Since $\mathrm{H}^2(\mathfrak{u}, k)_\gamma \neq 0$, we may apply Proposition 3.1(C) to conclude one of the following:

- (a) $\gamma = \alpha + \beta = i\beta_1 + \beta_2$ for $0 < i < p$, $\beta_1, \beta_2 \in \Delta$.

In this case, we can apply Lemma 2.5 to conclude that $\gamma = -(s_{\beta_1}s_{\beta_2}) \cdot 0$.

- (b) γ is a solution to (3.1.2).

This case has been dealt with.

- (c) $\gamma = \alpha + \beta$ is one of the exceptions listed in Proposition 3.1(C).

The solutions in (i) through (vii) of Proposition 3.1(C) are of the form $\alpha + \beta = -w \cdot 0$, for some $w \in W$ of length 2. That leaves the cases listed above. \square

Remark *Note that the excluded weights all have the form $\gamma = -w \cdot 0 + p\sigma$ for $w \in W$ with $l(w) = 2$ and $\sigma \in X(T)$. Indeed, we have (i) $\gamma = -(s_{\alpha_{n-1}}s_{\alpha_{n-2}}) \cdot 0 + 3\alpha_n$, (ii) $\gamma = -(s_{\alpha_{n-2}}s_n) \cdot 0 + 3\alpha_{n-1}$, (iii) $\gamma = -(s_{\alpha_2}s_{\alpha_1}) \cdot 0 + 3\alpha_3$ or $\gamma = -(s_{\alpha_2}s_{\alpha_4}) \cdot 0 + 3\alpha_3$.*

4.4 \mathfrak{u} -cohomology: Combining Proposition 4.3 with Corollary 2.3, one can identify precisely the T -structure of $\mathrm{H}^2(\mathfrak{u}, k)$ for $p \geq 3$. If $p = 3$ and Φ is of type G_2 , the results are obtained by direct computation. The answer is summarized in the following theorem. Notice that the answer is the same as in characteristic zero if $p > 3$ and in some cases when $p = 3$.

Theorem *Let $p \geq 3$ and $\pi = \{-w \cdot 0 : w \in W, l(w) = 2\}$. As a T -module*

$$\mathrm{H}^2(\mathfrak{u}, k) \cong \bigoplus_{\lambda \in \pi \cup \pi'} \lambda$$

where π' is given below. Further, if $\lambda = -w \cdot 0$ with $w = s_\alpha s_\beta$, then the corresponding cohomology class is represented by $\phi_\alpha \wedge \phi_{-\langle \beta, \alpha^\vee \rangle \alpha + \beta}$.

- (a) Suppose $p > 3$: $\pi' = \emptyset$.

- (b) Suppose $p = 3$ and Φ is of type A_n , $B_2 = C_2$, D_n , or E_n : $\pi' = \emptyset$.

(c) Suppose $p = 3$ and Φ is of type B_n , $n \geq 3$: $\pi' = \{\alpha_{n-2} + 2\alpha_{n-1} + 3\alpha_n\}$ corresponding to the cohomology class

$$\phi_{\alpha_n} \wedge \phi_{\alpha_{n-2}+2\alpha_{n-1}+2\alpha_n} - \phi_{\alpha_{n-1}+\alpha_n} \wedge \phi_{\alpha_{n-2}+\alpha_{n-1}+2\alpha_n} + \phi_{\alpha_{n-2}+\alpha_{n-1}+\alpha_n} \wedge \phi_{\alpha_{n-1}+2\alpha_n}.$$

(d) Suppose $p = 3$ and Φ is of type C_n , $n \geq 3$: $\pi' = \{\alpha_{n-2} + 3\alpha_{n-1} + \alpha_n\}$ corresponding to the cohomology class

$$\phi_{\alpha_{n-1}} \wedge \phi_{\alpha_{n-2}+2\alpha_{n-1}+\alpha_n} - \phi_{\alpha_{n-2}+\alpha_{n-1}} \wedge \phi_{2\alpha_{n-1}+\alpha_n}.$$

(e) Suppose $p = 3$ and Φ is of type F_4 : $\pi' = \{\alpha_1 + 2\alpha_2 + 3\alpha_3, \alpha_2 + 3\alpha_3 + \alpha_4\}$ corresponding to the cohomology classes

$$\phi_{\alpha_3} \wedge \phi_{\alpha_1+2\alpha_2+2\alpha_3} - \phi_{\alpha_2+\alpha_3} \wedge \phi_{\alpha_1+\alpha_2+2\alpha_3} + \phi_{\alpha_1+\alpha_2+\alpha_3} \wedge \phi_{\alpha_2+2\alpha_3},$$

$$\phi_{\alpha_3} \wedge \phi_{\alpha_2+2\alpha_3+\alpha_4} - \phi_{\alpha_3+\alpha_4} \wedge \phi_{\alpha_2+2\alpha_3}.$$

(f) Suppose $p = 3$ and Φ is of type G_2 : $\pi' = \{3\alpha_1+\alpha_2, 3\alpha_1+3\alpha_2, 6\alpha_1+3\alpha_2, 4\alpha_1+2\alpha_2\}$ corresponding to the cohomology classes

$$\phi_{\alpha_1} \wedge \phi_{2\alpha_1+\alpha_2}, \phi_{\alpha_2} \wedge \phi_{3\alpha_1+2\alpha_2}, \phi_{3\alpha_1+\alpha_2} \wedge \phi_{3\alpha_1+2\alpha_2},$$

$$\phi_{\alpha_1} \wedge \phi_{3\alpha_1+2\alpha_2} + \phi_{\alpha_1+\alpha_2} \wedge \phi_{3\alpha_1+\alpha_2}.$$

4.5 B -structure on $H^2(\mathfrak{u}, k)$: Theorem 4.4 identifies the T -structure of $H^2(\mathfrak{u}, k)$. With the aid of Theorem 5.3 below, we can further describe the B -structure. We include the discussion here for continuity. Of interest is the action of U . Let u be in U and $x \in H^2(\mathfrak{u}, k)$ have weight λ . Since U is generated by negative root subgroups, either $u \cdot x = 0$ or $u \cdot x$ has weight $\mu = \lambda + \sigma$ where $\sigma = \sum_{\alpha \in \Delta} c_\alpha \alpha$ with each $c_\alpha \leq 0$. In particular, under the standard order relation on weights, $\mu \leq \lambda$.

For $p > 3$, all weights of $H^2(\mathfrak{u}, k)$ are of the form $-w \cdot 0$ for $w \in W$ with $l(w) = 2$. However, there are no order relations between such weights and hence the U -action must be trivial. Hence, $H^2(\mathfrak{u}, k)$ is semi-simple as a B -module.

For $p = 3$, the same conclusion holds in types A_n , D_n , $B_2 = C_2$, and E_n where $\pi' = \emptyset$. In the remaining cases, one can have order relations involving the weights in π' . Consider the spectral sequence (4.1.1). From this, there is an exact sequence

$$0 \rightarrow (\mathfrak{u}^*)^{(1)} \rightarrow H^2(U_1, k) \rightarrow H^2(\mathfrak{u}, k) \rightarrow H^1(\mathfrak{u}, k) \otimes (\mathfrak{u}^*)^{(1)}.$$

Any element of $H^2(\mathfrak{u}, k)$ that is in the image of the map from $H^2(U_1, k)$ must be fixed by U_1 . Hence a root subgroup $U_{-\alpha} \subset U$ must shift such weights by a multiple of $p\alpha$. By Proposition 4.1, all elements having weight in π' must

be U_1 -fixed except in type G_2 for the weight $3\alpha_1 + \alpha_2$. However there are no weights smaller than $3\alpha_1 + \alpha_2$ in $H^2(\mathfrak{u}, k)$. So we only need to consider root orderings involving multiples of 3α for $\alpha \in \Phi$.

In type B_n ($n \geq 3$), there is only one such ordering: $-(s_{\alpha_{n-1}}s_{\alpha_{n-2}}) \cdot 0 = \alpha_{n-2} + 2\alpha_{n-1}$ and $\alpha_{n-2} + 2\alpha_{n-1} + 3\alpha_n$. Let $\pi'' = \pi - \{\alpha_{n-2} + 2\alpha_{n-1}\}$. Then, as a B -module,

$$H^2(\mathfrak{u}, k) \cong M \oplus \bigoplus_{\lambda \in \pi''} \lambda,$$

where M has factors $\alpha_{n-2} + 2\alpha_{n-1} + 3\alpha_n$ and $\alpha_{n-2} + 2\alpha_{n-1}$. The module M is in fact indecomposable. By Theorem 5.3(c)(xi), $(H^2(U_1, k) \otimes (s_{\alpha_{n-1}}s_{\alpha_{n-2}}) \cdot 0)^{T_1} \cong H^2(B_1, (s_{\alpha_{n-1}}s_{\alpha_{n-2}}) \cdot 0)$ is a two-dimensional indecomposable B -module. Hence, $H^2(U_1, k)$ must contain a subquotient that is a two-dimensional indecomposable B -module. Further, the image of this subquotient under the map $H^2(U_1, k) \rightarrow H^2(\mathfrak{u}, k)$ remains indecomposable and must be M . Note that the factors above are listed from top to bottom and $M \cong N_{B_n}^{(1)} \otimes (\alpha_{n-2} + 2\alpha_{n-1})$ where N_{B_n} is defined in Section 6.2.

In type C_n ($n \geq 3$), there is similarly one such ordering: $-(s_{\alpha_{n-2}}s_{\alpha_n}) \cdot 0 = \alpha_{n-2} + \alpha_n$ and $\alpha_{n-2} + 3\alpha_{n-1} + \alpha_n$. Let $\pi'' = \pi - \{\alpha_{n-2} + \alpha_n\}$. Then, as a B -module,

$$H^2(\mathfrak{u}, k) \cong M \oplus \bigoplus_{\lambda \in \pi''} \lambda,$$

where M is an indecomposable B -module with factors $\alpha_{n-2} + 3\alpha_{n-1} + \alpha_n$ and $\alpha_{n-2} + \alpha_n$. See Theorem 5.3(c)(xii). Note that $M \cong N_{C_n}^{(1)} \otimes (\alpha_{n-2} + \alpha_n)$.

In type F_4 , one gets two pairs (corresponding to the B_3 and C_3 Levi factors): $-(s_{\alpha_2}s_{\alpha_1}) \cdot 0 = \alpha_1 + 2\alpha_2$ and $\alpha_1 + 2\alpha_2 + 3\alpha_3$; $-(s_{\alpha_2}s_{\alpha_4}) \cdot 0 = \alpha_2 + \alpha_4$ and $\alpha_2 + 3\alpha_3 + \alpha_4$. Let $\pi'' = \pi - \{\alpha_1 + 2\alpha_2, \alpha_2 + \alpha_4\}$. Then, as a B -module,

$$H^2(\mathfrak{u}, k) \cong M_1 \oplus M_2 \oplus \bigoplus_{\lambda \in \pi''} \lambda,$$

where M_1 is an indecomposable B -module with factors $\alpha_1 + 2\alpha_2 + 3\alpha_3$ and $\alpha_1 + 2\alpha_2$, and M_2 is an indecomposable B -module with factors $\alpha_2 + 3\alpha_3 + \alpha_4$ and $\alpha_2 + \alpha_4$. See Theorem 5.3(c)(xiii, xiv). Note that $M_1 \cong N_{F_4}^{(1)} \otimes (\alpha_1 + 2\alpha_2)$ and $M_2 \cong N_{F_4}^{(1)} \otimes (\alpha_2 + \alpha_4)$.

In type G_2 , there are again two pairs: $-(s_{\alpha_2}s_{\alpha_1}) \cdot 0 = \alpha_1 + 2\alpha_2$ and $4\alpha_1 + 2\alpha_2$; $3\alpha_1 + 3\alpha_2$ and $6\alpha_1 + 3\alpha_2$. As a B -module,

$$H^2(\mathfrak{u}, k) \cong M_1 \oplus M_2 \oplus (4\alpha_1 + \alpha_2) \oplus (3\alpha_1 + \alpha_2),$$

where M_1 is an indecomposable B -module with factors $4\alpha_1 + 2\alpha_2$ and $\alpha_1 + 2\alpha_2$, and M_2 is an indecomposable B -module with factors $6\alpha_1 + 3\alpha_2$ and $3\alpha_1 + 3\alpha_2$. The indecomposability of M_1 follows from the indecomposability of $H^2(B_1, (s_{\alpha_2}s_{\alpha_1}) \cdot 0)$ (Theorem 5.3(c)(xv)) while the indecomposability of M_2

follows from the indecomposability of the two-dimensional module appearing in the description of $H^2(B_1, k)$ (Theorem 5.3(a)). Note that $M_1 \cong N_{G_2}^{(1)} \otimes (\alpha_1 + 2\alpha_2)$ and $M_2 \cong N_{G_2}^{(1)} \otimes (3\alpha_1 + 3\alpha_2)$.

5 B_r -cohomology

5.1 In this section, we compute $H^2(B_r, \lambda)$ for all $\lambda \in X(T)$. We recall that the first cohomology groups $H^1(B_1, \lambda)$ were computed for all primes and all weights $\lambda \in X(T)$ by Jantzen [Jan2]. For higher r , $H^1(B_r, \lambda)$ is computed by the authors in [BNP]. We investigate the second cohomology $H^2(B_r, \lambda)$ by starting out with B_1 .

Note that for $\lambda \in X(T)$, we may write $\lambda = \lambda_0 + p\lambda_1$ for unique weights λ_0, λ_1 with $\lambda_0 \in X_1(T)$. Then for all $i \geq 0$,

$$H^2(B_1, \lambda) = H^2(B_1, \lambda_0 + p\lambda_1) \cong H^2(B_1, \lambda_0) \otimes p\lambda_1.$$

Hence it suffices to compute $H^2(B_1, \lambda)$ for $\lambda \in X_1(T)$.

From Propositions 4.2 and 4.3, if $\lambda \neq 0$ and $H^2(B_1, \lambda) \neq 0$, then λ will usually (in fact always, as we will see) have the form $\lambda = w \cdot 0 + p\sigma$ for some $w \in W$ with $l(w) = 2$ and $\sigma \in X(T)$. Since we are focusing on restricted weights, given such a $w \in W$, we want to identify the unique weight $\sigma \in X(T)$ such that $\lambda = w \cdot 0 + p\sigma \in X_1(T)$. Such weights are summarized in the following lemma whose verification is straightforward and left to the interested reader.

Lemma *Let $p \geq 3$. For $w = s_{\alpha_i} s_{\alpha_j} \in W$ with $i \neq j$ we define*

$$\gamma_w = \begin{cases} \omega_i & \text{if } \alpha_i \text{ and } \alpha_j \text{ are adjacent roots in } \Phi. \\ \omega_i + \omega_j & \text{if } \alpha_i \text{ and } \alpha_j \text{ are not adjacent in } \Phi. \end{cases}$$

except in the following $p = 3$ cases, where we define

$$\gamma_w = \begin{cases} 2\omega_n & \text{if } \Phi \text{ of type } B_n, \text{ and } w = s_{\alpha_n} s_{\alpha_{n-1}}. \\ \omega_{n-1} - \omega_n & \text{if } \Phi \text{ of type } B_n, \text{ and } w = s_{\alpha_{n-1}} s_{\alpha_{n-2}}. \\ 2\omega_{n-1} - \omega_{n-2} & \text{if } \Phi \text{ of type } C_n, n \geq 3, \text{ and } w = s_{\alpha_{n-1}} s_{\alpha_n}. \\ \omega_{n-2} - \omega_{n-1} + \omega_n & \text{if } \Phi \text{ of type } C_n, n \geq 3, \text{ and } w = s_{\alpha_n} s_{\alpha_{n-2}}. \\ \omega_2 - \omega_3 & \text{if } \Phi \text{ of type } F_4, \text{ and } w = s_{\alpha_2} s_{\alpha_1}. \\ 2\omega_3 - \omega_4 & \text{if } \Phi \text{ of type } F_4, \text{ and } w = s_{\alpha_3} s_{\alpha_2}. \\ \omega_2 - \omega_3 + \omega_4 & \text{if } \Phi \text{ of type } F_4, \text{ and } w = s_{\alpha_2} s_{\alpha_4}. \\ 2\omega_1 & \text{if } \Phi \text{ of type } G_2, \text{ and } w = s_{\alpha_1} s_{\alpha_2}. \\ \omega_2 - \omega_1 & \text{if } \Phi \text{ of type } G_2, \text{ and } w = s_{\alpha_2} s_{\alpha_1}. \end{cases}$$

Then $w \cdot 0 + p\gamma_w \in X_1(T)$.

5.2 The following lemma will allow us to deal with the exceptional weights appearing in Proposition 4.1 and the “extra” or non-generic weights appearing in Theorem 4.4.

Lemma Suppose $p = 3$.

- (a) $\text{Hom}_{B/B_1}(k, H^2(B_1, \lambda)) = 0$ for each of the following
- (i) Φ is of type B_n ($n \geq 3$) and $\lambda = -\alpha_{n-2} - 2\alpha_{n-1} - 3\alpha_n$,
 - (ii) Φ is of type C_n ($n \geq 3$) and $\lambda = -\alpha_{n-2} - 3\alpha_{n-1} - \alpha_n$,
 - (iii) Φ is of type F_4 and $\lambda = -\alpha_1 - 2\alpha_2 - 3\alpha_3$ or $\lambda = -\alpha_2 - 3\alpha_3 - \alpha_4$,
 - (iv) Φ is of type G_2 and $\lambda = -4\alpha_1 - 2\alpha_2$.
- (b) $H^2(B_1, \lambda) = 0$ for each of the following
- (i) Φ is of type B_n ($n \geq 3$) and $\lambda = -\alpha_{n-1} - 3\alpha_n = s_{\alpha_n} s_{\alpha_{n-1}} \cdot 0$,
 - (ii) Φ is of type C_n ($n \geq 3$) and $\lambda = -3\alpha_{n-1} - \alpha_n = s_{\alpha_{n-1}} s_{\alpha_n} \cdot 0$,
 - (iii) Φ is of type F_4 and $\lambda = -\alpha_2 - 3\alpha_3 = s_{\alpha_3} s_{\alpha_2} \cdot 0$,
 - (iv) Φ is of type G_2 and $\lambda = -3\alpha_1 - \alpha_2$.

Proof. We first identify $H^2(B, \lambda)$ for the above weights. For the weights listed in (b) (i)-(iii), we have $\lambda = w \cdot 0$ for some $w \in W$ with $l(w) = 2$. Therefore, by [And, 2.2] $H^2(B, \lambda) \cong k$.

For each weight λ in (a)(i)-(iv) and (b)(iv), we will show that $H^2(B, \lambda) = 0$. For each of these weights, there exists a unique simple root α such that $\langle \lambda, \alpha^\vee \rangle = -3$ or -2 . Consider the spectral sequence (cf. [Jan1, I.4.5(b)])

$$E_2^{i,j} = H^i(P(\alpha), R^j \text{ind}_B^{P(\alpha)} \lambda) \Rightarrow H^{i+j}(B, \lambda).$$

By [Jan1, II.5.2(d)], $R^j \text{ind}_B^{P(\alpha)} \lambda = 0$ for all $j \neq 1$ and so $E_2^{i,j} = 0$ for all $j \neq 1$. Hence the spectral sequence collapses to give

$$H^{i+1}(B, \lambda) \cong H^i(P(\alpha), R^1 \text{ind}_B^{P(\alpha)} \lambda)$$

for all i . In particular, $H^2(B, \lambda) \cong H^1(P(\alpha), R^1 \text{ind}_B^{P(\alpha)} \lambda)$. Observe that $\langle s_\alpha \cdot \lambda, \alpha^\vee \rangle = -\langle \lambda, \alpha^\vee \rangle - 2 = 1$ or 0 respectively. Applying [Jan1, II.5.3(b)] to $s_\alpha \cdot \lambda$, we have $\text{ind}_B^{P(\alpha)} s_\alpha \cdot \lambda \cong R^1 \text{ind}_B^{P(\alpha)} \lambda$. Using Frobenius reciprocity, we then have

$$H^2(B, \lambda) \cong H^1(P(\alpha), R^1 \text{ind}_B^{P(\alpha)} \lambda) \cong H^1(P(\alpha), \text{ind}_B^{P(\alpha)} s_\alpha \cdot \lambda) \cong H^1(B, s_\alpha \cdot \lambda).$$

Since $s_\alpha \cdot \lambda \neq -p^m \beta$ for $m \geq 0$ and $\beta \in \Delta$, by [And, 2.4], $H^2(B, \lambda) \cong H^1(B, s_\alpha \cdot \lambda) = 0$.

Summarizing the above, we have

$$H^2(B, \lambda) \cong \begin{cases} 0 & \text{for } \lambda \text{ listed in (a) or (b)(iv)} \\ k & \text{for } \lambda \text{ listed in (b)(i)-(iii)}. \end{cases} \quad (5.2.1)$$

We now compute $\text{Hom}_{B/B_1}(k, H^2(B_1, \lambda))$ in all cases by using the LHS spectral sequence

$$E_2^{i,j} = H^i(B/B_1, H^j(B_1, \lambda)) \Rightarrow H^{i+j}(B, \lambda).$$

Notice that $\text{Hom}_{B/B_1}(k, H^2(B_1, \lambda)) = E_2^{0,2}$. Also, for each of the weights listed in (a) and (b), since $\lambda \not\equiv 0 \pmod{pX(T)}$, we have $E_2^{i,0} = 0$ for $i \geq 0$.

Next we identify the term $E_2^{i,1} = H^i(B/B_1, H^1(B_1, \lambda))$. For the weights listed in part (a) one obtains from [Jan2, 3.2] that $H^1(B_1, \lambda) = 0$ and so $E_2^{i,1} = 0$ for all i . For the weights listed in (b) (i)-(iii), it follows from [Jan2, 3.5] that $H^1(B_1, \lambda)$ is of the form $-p\alpha$ for some $\alpha \in \Delta$. Then $E_2^{i,1} = H^i(B/B_1, -p\alpha) \cong H^i(B, -\alpha)^{(1)}$. By [And, 2.2], $E_2^{i,1} = 0$ for $i \neq 1$, while $E_2^{1,1} = k$. In the case (b) (iv), one obtains from [Jan2, 3.7] that $H^1(B_1, \lambda)$ has a filtration with factors k and $-p\alpha_1$. Again using [And, 2.2] one gets that $E_2^{i,1} = 0$ for $i \neq 1$. (The argument below will show that $E_2^{1,1} = 0$ also but it is not necessary to know this at this point in the argument.)

Summarizing the above, we have:

$$E_2^{i,1} = \begin{cases} 0 & \text{for all } i \neq 1 \text{ and all } \lambda \text{ listed} \\ k & \text{for } i = 1 \text{ and } \lambda \text{ listed in (b)(i)-(iii)}. \end{cases} \quad (5.2.2)$$

In all cases, we have seen that $E_2^{i,0} = 0$ for all i and $E_2^{i,1} = 0$ for all $i \neq 1$. Hence, the terms $E_2^{0,2}$ and $E_2^{1,1}$ consist of universal cycles and so as T -modules the abutment

$$H^2(B, \lambda) \cong E_2^{1,1} \oplus E_2^{0,2}.$$

For those weights λ listed in (a), it follows immediately from (5.2.1) that $\text{Hom}_{B/B_1}(k, H^2(B_1, \lambda)) = E_2^{0,2} = 0$ as claimed. It can also be seen that $\text{Hom}_{B/B_1}(k, H^2(B_1, \lambda)) = 0$ for those λ listed in part (b). The case (b)(iv) follows exactly as for part (a). For the weights in (b)(i)-(iii), by (5.2.1) and (5.2.2), we see that $H^2(B, \lambda) \cong k \cong E_2^{1,1}$. Hence $E_2^{0,2} = 0$.

For those λ listed in part (b), we now consider $H^2(B_1, \lambda)$. By Proposition 4.2, as T -modules,

$$H^2(B_1, \lambda) \subset \bigoplus_{\nu \in X(T)} p\nu^{\dim H^2(\mathbf{u}, k) - \lambda + p\nu}.$$

By Theorem 4.4, $\dim H^2(\mathbf{u}, k)_{-\lambda} = 1$. Further, for $\nu \neq 0$, by Theorem 4.4 and Proposition 3.1(C), $\dim H^2(\mathbf{u}, k)_{-\lambda + p\nu} = 0$. Hence, $H^2(B_1, \lambda) \subset k$ as

a T -module (and hence also as a B -module). If $H^2(B_1, \lambda) = k$, then we also have $\text{Hom}_{B/B_1}(k, H^2(B_1, \lambda)) = k$. However, we have seen above that $\text{Hom}_{B/B_1}(k, H^2(B_1, \lambda)) = 0$. Hence $H^2(B_1, \lambda) = 0$ for the weights λ listed in part (b). \square

5.3 B_1 -cohomology: The following theorem describes the H^2 -cohomology for B_1 as a rational B -module when $p \geq 3$ and $\lambda \in X_1(T)$. Recall that $B \cong B^{(1)} \cong B/B_1$ and from Section 4.1 that for an arbitrary weight $\lambda \in X(T)$, one has $H^2(B_1, \lambda) \cong H^2(B_1, \lambda_0) \otimes p\lambda_1$ for unique weights $\lambda_0 \in X_1(T)$, $\lambda_1 \in X(T)$.

Theorem *Let $p \geq 3$ and $\lambda \in X_1(T)$. Then the following isomorphisms hold as B -modules.*

- (a) $H^2(B_1, k) \cong \mathfrak{u}^{*(1)}$ except in the following cases:
 (i) $p = 3$, Φ is of type A_2 , where

$$H^2(B_1, k) \cong \mathfrak{u}^{*(1)} \oplus \omega_1^{(1)} \oplus \omega_2^{(1)}.$$

- (ii) $p = 3$, Φ is of type G_2 , where

$H^2(B_1, k)$ is a non-split extension of the indecomposable B -module with factors $\omega_1^{(1)}$ and $(\omega_2 - \omega_1)^{(1)}$ by $\mathfrak{u}^{*(1)}$.

- (b) If $\lambda \neq 0$ and $H^2(B_1, \lambda) \neq 0$, then $\lambda = w \cdot 0 + p\gamma_w$ for some $w \in W$ with $l(w) = 2$.
 (c) If $\lambda = w \cdot 0 + p\gamma_w$ for some $w \in W$ with $l(w) = 2$ and $\lambda \neq 0$, then

$$H^2(B_1, w \cdot 0 + p\gamma_w) \cong \gamma_w^{(1)},$$

except in the following cases:

- (i) $p = 5$, Φ is of type A_4 , and $w \in \{s_{\alpha_2}s_{\alpha_1}, s_{\alpha_3}s_{\alpha_4}\}$, where

$$H^2(B_1, w \cdot 0 + p\gamma_w) = H^2(B_1, 2\omega_2 + 2\omega_3) \cong \omega_2^{(1)} \oplus \omega_3^{(1)}.$$

- (ii) $p = 3$, Φ is of type A_5 , and $w \in \{s_{\alpha_1}s_{\alpha_2}, s_{\alpha_5}s_{\alpha_4}\}$, where

$$H^2(B_1, w \cdot 0 + p\gamma_w) = H^2(B_1, \omega_3) \cong \omega_1^{(1)} \oplus \omega_5^{(1)}.$$

- (iii) $p = 3$, Φ is of type A_5 , and $w \in \{s_{\alpha_2}s_{\alpha_1}, s_{\alpha_4}s_{\alpha_5}\}$, where

$$H^2(B_1, w \cdot 0 + p\gamma_w) = H^2(B_1, 2\omega_3) \cong \omega_2^{(1)} \oplus \omega_4^{(1)}.$$

- (iv) $p = 3$, Φ is of type A_5 , and $w \in \{s_{\alpha_1}s_{\alpha_4}, s_{\alpha_2}s_{\alpha_5}\}$, where

$$H^2(B_1, w \cdot 0 + p\gamma_w) = H^2(B_1, \rho) \cong (\omega_1 + \omega_4)^{(1)} \oplus (\omega_2 + \omega_5)^{(1)}.$$

(v) $p = 3$, Φ is of type E_6 , and $w \in \{s_{\alpha_1}s_{\alpha_3}, s_{\alpha_6}s_{\alpha_5}\}$, where

$$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma_w) = \mathrm{H}^2(B_1, \omega_4) \cong \omega_1^{(1)} \oplus \omega_6^{(1)}.$$

(vi) $p = 3$, Φ is of type E_6 , and $w \in \{s_{\alpha_3}s_{\alpha_1}, s_{\alpha_5}s_{\alpha_6}\}$, where

$$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma_w) = \mathrm{H}^2(B_1, 2\omega_4) \cong \omega_3^{(1)} \oplus \omega_5^{(1)}.$$

(vii) $p = 3$, Φ is of type E_6 , and $w \in \{s_{\alpha_1}s_{\alpha_5}, s_{\alpha_3}s_{\alpha_6}\}$, where

$$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma_w) = \mathrm{H}^2(B_1, \rho - \omega_2) \cong (\omega_1 + \omega_5)^{(1)} \oplus (\omega_3 + \omega_6)^{(1)}.$$

(viii) $p = 3$, Φ is of type B_n , $n \geq 3$, and $w = s_{\alpha_n}s_{\alpha_{n-1}}$, where

$$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma_w) = \mathrm{H}^2(B_1, \omega_{n-2} + \omega_{n-1} + 2\omega_n) = 0.$$

(ix) $p = 3$, Φ is of type C_n , and $w = s_{\alpha_{n-1}}s_{\alpha_n}$, where

$$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma_w) = \mathrm{H}^2(B_1, 2\omega_{n-1} + \omega_n) = 0.$$

(x) $p = 3$, Φ is of type F_4 , and $w = s_{\alpha_3}s_{\alpha_2}$, where

$$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma_w) = \mathrm{H}^2(B_1, \omega_2 + 2\omega_3) = 0.$$

(xi) $p = 3$, Φ is of type B_n , $n \geq 3$, and $w = s_{\alpha_{n-1}}s_{\alpha_{n-2}}$, where

$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma_w) = \mathrm{H}^2(B_1, \omega_{n-3} + \omega_n)$ is an indecomposable B -module with factors $\omega_n^{(1)}$ and $(\omega_{n-1} - \omega_n)^{(1)}$.

(xii) $p = 3$, Φ is of type C_n , $n \geq 3$, and $w = s_{\alpha_{n-2}}s_{\alpha_n}$, where

$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma_w) = \mathrm{H}^2(B_1, \omega_{n-3} + \omega_{n-2} + \omega_n)$ is an indecomposable B -module with factors $\omega_{n-1}^{(1)}$ and $(\omega_{n-2} - \omega_{n-1} + \omega_n)^{(1)}$.

(xiii) $p = 3$, Φ is of type F_4 , and $w = s_{\alpha_2}s_{\alpha_1}$, where

$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma_w) = \mathrm{H}^2(B_1, \omega_3)$ is an indecomposable B -module with factors $(\omega_3 - \omega_4)^{(1)}$ and $(\omega_2 - \omega_3)^{(1)}$.

(xiv) $p = 3$, Φ is of type F_4 , and $w = s_{\alpha_2}s_{\alpha_4}$, where

$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma_w) = \mathrm{H}^2(B_1, \omega_1 + \omega_2 + \omega_4)$ is an indecomposable B -module with factors $\omega_3^{(1)}$ and $(\omega_2 - \omega_3 + \omega_4)^{(1)}$.

(xv) $p = 3$, Φ is of type G_2 , and $w = s_{\alpha_2}s_{\alpha_1}$, where

$\mathrm{H}^2(B_1, w \cdot 0 + p\gamma_w) = \mathrm{H}^2(B_1, \omega_1)$ is an indecomposable B -module with factors $\omega_1^{(1)}$ and $(\omega_2 - \omega_1)^{(1)}$.

Proof. (a) If $p = 3$ and Φ is of type G_2 the assertion can be found in [AJ, 6.12]. In the remaining cases we conclude from (4.1.2) and (4.1.4) that

$H^2(B_1, k)^{(-1)} \cong \mathfrak{u}^*$, unless $H^2(\mathfrak{u}, k)^{T_1} \neq 0$. In which case we conclude from Corollary 2.4(a) that there exist $\alpha \in \Delta$ and $\beta \in \Phi^+$ with $\alpha \neq \beta$ and $\alpha + \beta = p\sigma$ for some $\sigma \in X(T)$, i.e. $\alpha + \beta$ is a solution of equation (3.1.1). Now Proposition 3.1(A) implies that $p = 3$ and Φ is of type A_2 . The assertion follows immediately from (4.1.4) and the fact that the weights ω_1, ω_2 , and zero are not in the same coset of the root lattice (see also [AJ, 6.2]).

(b) Assume that $\lambda \neq 0$ and $H^2(B_1, \lambda) \neq 0$. If $p = 3$ assume further that Φ is not of type G_2 . By Proposition 4.2, λ is either of the desired form or there exists a $\mu \in X(T)$ with $\lambda \equiv -\mu \pmod{pX(T)}$ and $H^2(\mathfrak{u}, k)_\mu \neq 0$. It follows from Proposition 4.3 that either $\mu = -w \cdot 0$ with $l(w) = 2$, or $\mu \equiv -w \cdot 0 \pmod{pX(T)}$ with $l(w) = 2$ (see Remark 4.3).

If $p = 3$ and Φ is of type G_2 , then Theorem 4.4(f) implies that $\mu = -w \cdot 0$ with $l(w) = 2$, unless $\mu = 3\alpha_1 + \alpha_2$ (note that $4\alpha_1 + 2\alpha_2 = -s_{\alpha_2}s_{\alpha_1} \cdot 0 + 3\alpha_1$). The assertion follows now from Lemma 5.2 (b)(iv).

(c) Cases (viii) through (x) have been dealt with in Lemma 5.2(b). One might refer to these exceptional cases as “vanishing” cohomology classes. Moreover, from (b) we know that $H^2(B_1, -3\alpha_1 - \alpha_2) = 0$ for $p = 3$ and Φ being of type G_2 . We may therefore exclude these weights from further discussion and apply Proposition 4.2 and Theorem 4.4 to the remaining cases. One concludes that $H^2(B_1, p\gamma_w + w \cdot 0)^{(-1)} \cong \gamma_w$ unless $-w \cdot 0$ is congruent modulo $pX(T)$ to another weight space of $H^2(\mathfrak{u}, k)$.

First we consider the possibility that $w_1 \cdot 0 \equiv w_2 \cdot 0 \pmod{pX(T)}$ for distinct $w_1, w_2 \in W$, both of length 2. This implies that the weight $-w_1 \cdot 0$ is a non-trivial solution to equation (3.1.3). From Proposition 3.1(C) one obtains cases (i) through (vii). These exceptions result in a “pairing” or “doubling” of cohomology. Since the weights involved are not in the same coset of the root-lattice we obtain direct sums of two weight spaces. Note that the solutions to equations (3.1.1) and (3.1.2) have been dealt with.

Finally we have to consider those cases where some $w \cdot 0$ is congruent modulo $pX(T)$ to one of the additional cohomology classes listed in Theorem 4.4. These correspond to the remaining exceptions (xi) through (xv). In each of these cases one obtains from Proposition 4.2 that $H^2(B_1, w \cdot 0)$ is two-dimensional with weights k and $p\alpha$, where $\alpha \in \Delta$. In each of these four cases it follows from Lemma 5.2(a) that $\text{Hom}_B(p\alpha, H^2(B_1, w \cdot 0)) = \text{Hom}_B(k, H^2(B_1, w \cdot 0 - p\alpha)) = 0$. Thus each $H^2(B_1, w \cdot 0)$ is an indecomposable B -module. The assertion follows now from Lemma 5.1. \square

Remark *For the indecomposable B -modules appearing in the preceding theorem, the factors are listed from top to bottom. That is, the first factor is the head and the second factor is the socle. Also, in parts (xi) and (xii) when*

$n = 3$, ω_0 is defined to be zero.

Corollary Let $p \geq 3$ and $\lambda, \gamma \in X(T)$.

- (a) If $\lambda \notin pX(T)$ and $\lambda \neq w \cdot 0 + p\sigma$ for some $w \in W$ with $l(w) = 2$ and $\sigma \in X(T)$, then $H^2(B_1, \lambda) = 0$.
(b) If $\alpha \in \Delta$, then $H^2(B_1, p\gamma - \alpha) = 0$.

5.4 B_r -cohomology: We next consider $H^2(B_r, \lambda)$ for higher r . The following observation will be used in several succeeding results.

Lemma Let $p \geq 3$, $\lambda \in X(T)$, and $1 \leq l < r$.

- (a) If $p = 3$ suppose that Φ is not of type A_2 . Then

$$\begin{aligned} \text{Hom}_{B_r/B_l}(k, H^2(B_1, k)^{(l-1)} \otimes p^l \lambda) \\ \cong \begin{cases} p^r \gamma & \text{if } \lambda = p^{r-l} \gamma - \alpha, \alpha \in \Delta, \gamma \in X(T), \\ 0 & \text{else .} \end{cases} \end{aligned}$$

- (b) Suppose $p = 3$ and Φ is of type A_2 . Then

$$\begin{aligned} \text{Hom}_{B_r/B_l}(k, H^2(B_1, k)^{(l-1)} \otimes p^l \lambda) \\ \cong \begin{cases} p^r \gamma & \text{if } \lambda = p^{r-l} \gamma - \alpha, \alpha \in \Delta, \gamma \in X(T), \\ p^r \gamma & \text{if } \lambda = p^{r-l} \gamma - \omega_i, i \in \{1, 2\}, \gamma \in X(T) \\ 0 & \text{else .} \end{cases} \end{aligned}$$

Proof. We have

$$\begin{aligned} \text{Hom}_{B_r/B_l}(k, H^2(B_1, k)^{(l-1)} \otimes p^l \lambda) &\cong \text{Hom}_{B_{r-l}}(k, H^2(B_1, k)^{(-1)} \otimes \lambda)^{(l)} \\ &\cong \text{Hom}_{B_{r-l}}(-\lambda, H^2(B_1, k)^{(-1)})^{(l)}. \end{aligned}$$

Now, $H^2(B_1, k)^{(-1)}$ is given by Theorem 5.3(a). We only need to consider the B -socle of $H^2(B_1, k)^{(-1)}$. Except for type A_2 when $p = 3$, the B -socle of $H^2(B_1, k)^{(-1)}$ is the B -socle of \mathfrak{u}^* . By [Jan2, 2.2, 2.4], the B -socle of \mathfrak{u}^* is $\bigoplus_{\beta \in \Delta} k_\beta$ and part (a) follows. For part (b), the B -socle of $H^1(B_1, k)^{(-1)}$ also contains $\omega_1 \oplus \omega_2$ which yields the additional weights. \square

The following proposition gives a recursive formula for computing $H^2(B_r, k)$ from $H^2(B_1, k)$.

Proposition Let $p \geq 3$. Then $H^2(B_r, k) \cong H^2(B_1, k)^{(r-1)}$.

Proof. We use induction on r . Assume that $r > 1$. We use the LHS spectral

sequence

$$E_2^{i,j} = H^i(B_r/B_1, H^j(B_1, k)) \Rightarrow H^{i+j}(B_r, k).$$

Since p is odd one has $E_2^{i,1} = 0$ [And]. From Lemma 5.4 (with $l = 1$ and $\lambda = 0$), $E_2^{0,2} = \text{Hom}_{B_r/B_1}(k, H^2(B_1, k)) = 0$. It follows from the induction hypothesis that $H^2(B_r, k) \cong E_2^{2,0} = H^2(B_r/B_1, k) \cong H^2(B_{r-1}, k)^{(1)} \cong H^2(B_1, k)^{(r-1)}$. \square

5.5 To compute $H^2(B_r, \lambda)$ for $\lambda \notin p^r X(T)$, we begin with some special computations in type G_2 when $p = 3$. We define N_{G_2} to be the two-dimensional indecomposable B -module with factors α_1 and k (from top to bottom). Notice that it follows from [Jan2, 3.7] that $N_{G_2} \cong H^1(B_1, -\alpha_2)^{(-1)}$ and from Theorem 5.3 that $N_{G_2} \cong H^2(B_1, (s_{\alpha_2} s_{\alpha_1}) \cdot 0)^{(-1)}$. Moreover, $N_{G_2} \otimes \lambda$ remains indecomposable for any weight λ .

Lemma (A) *Suppose $p = 3$ and Φ is of type G_2 . Then as B -modules*

$$H^1(U_1, N_{G_2}) \cong H^1(\mathfrak{u}, N_{G_2})$$

and $H^1(U_1, N_{G_2})$ has a basis of T -eigenvectors $\{\alpha_2, 3\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1\}$.

Proof. By direct computation, one finds that $H^1(\mathfrak{u}, N_{G_2})$ has a basis of T -eigenvectors as given. The computation is similar to that for $H^1(\mathfrak{u}, k)$ as given in Section 2.1 (see [Jan1, I.9.17]) but involves a non-trivial map $d_0 : N_{G_2} \rightarrow N_{G_2} \otimes \mathfrak{u}^*$ along with $d_1 : N_{G_2} \otimes \mathfrak{u}^* \rightarrow N_{G_2} \otimes \Lambda^2 \mathfrak{u}^*$.

Now consider the spectral sequence (4.1.1). By [FP3], this can be modified to

$$E_2^{2i,j} = S^i(\mathfrak{u}^*)^{(1)} \otimes H^j(\mathfrak{u}, N_{G_2}) \Rightarrow H^{2i+j}(U_1, N_{G_2}).$$

Consider the differential $d_2 : E_2^{0,1} = H^1(\mathfrak{u}, N_{G_2}) \rightarrow (\mathfrak{u}^*)^{(1)} = E_2^{2,0}$. Considering the weights of $H^1(\mathfrak{u}, N_{G_2})$, we see that this map must be zero. Hence $H^1(U_1, N_{G_2}) \cong H^1(\mathfrak{u}, N_{G_2})$ as claimed. \square

Lemma (B) *Suppose $p = 3$, Φ is of type G_2 , and $\lambda \in X(T)$. Then $H^1(B_r, N_{G_2} \otimes \lambda)^{(-r)} \cong$*

$$\begin{cases} \gamma & \text{if } \lambda = p^r \gamma - p^l \alpha, \text{ with } 0 \leq l < r \text{ and } \alpha \in \Delta, \\ & \text{where } l \neq r-1 \text{ if } \alpha = \alpha_2, \text{ and } l \neq 0 \text{ if } \alpha = \alpha_1, \\ \gamma & \text{if } \lambda = p^r \gamma - (\alpha + \alpha_1), \text{ with } \alpha \in \Delta, \\ N_{G_2} \otimes \gamma & \text{if } \lambda = p^r \gamma - p^{r-1} \alpha_2, \\ 0 & \text{else.} \end{cases}$$

Proof. We consider first the case $r = 1$. Note that $H^1(B_1, N_{G_2} \otimes \lambda) \cong H^1(U_1, N_{G_2} \otimes \lambda)^{T_1} \cong (H^1(U_1, N_{G_2}) \otimes \lambda)^{T_1}$. From Lemma 5.5(A), this is zero

unless $\lambda = p\gamma - \alpha_2$, $\lambda = p\gamma - (\alpha_1 + \alpha_2)$, or $\lambda = p\gamma - 2\alpha_1$ for some $\gamma \in X(T)$. In the last two cases, one immediately gets that $H^1(B_1, N_{G_2} \otimes \lambda) \cong p\gamma$.

In the first case ($\lambda = p\gamma - \alpha_2$), $H^1(B_1, N_{G_2} \otimes \lambda)$ will be two-dimensional with a basis of T -eigenvectors being $\{p\gamma, p(\gamma + \alpha_1)\}$. To identify the B -structure, consider the short exact sequence of B -modules

$$0 \rightarrow \lambda \rightarrow N_{G_2} \otimes \lambda \rightarrow \alpha_1 + \lambda \rightarrow 0$$

and a portion of the associated long exact sequence (of B -modules) in cohomology:

$$\begin{aligned} \cdots \rightarrow \text{Hom}_{B_1}(k, \alpha_1 + p\gamma - \alpha_2) &\rightarrow H^1(B_1, p\gamma - \alpha_2) \\ &\rightarrow H^1(B_1, N_{G_2} \otimes \lambda) \rightarrow H^1(B_1, \alpha_1 + p\gamma - \alpha_2) \rightarrow \cdots \end{aligned}$$

The Hom-group is evidently zero. Also, by [Jan2, 3.2], $H^1(B_1, \alpha_1 + p\gamma - \alpha_2) = 0$. Hence, $H^1(B_1, N_{G_2} \otimes \lambda) \cong H^1(B_1, p\gamma - \alpha_2) \cong N_{G_2}^{(1)} \otimes p\gamma$ where the last isomorphism follows from [Jan2, 3.2] as noted above.

We summarize for $r = 1$:

$$H^1(B_1, N_{G_2} \otimes \lambda)^{(-1)} \cong \begin{cases} \gamma & \text{if } \lambda = p\gamma - (\alpha_1 + \alpha_2), \\ \gamma & \text{if } \lambda = p\gamma - 2\alpha_1, \\ N_{G_2} \otimes \gamma & \text{if } \lambda = p\gamma - \alpha_2, \\ 0 & \text{else.} \end{cases} \quad (5.5.1)$$

For $r > 1$ we consider the LHS spectral sequence

$$E_2^{i,j} = H^i(B_r/B_1, H^j(B_1, N_{G_2} \otimes \lambda)) \Rightarrow H^{i+j}(B_r, N_{G_2} \otimes \lambda).$$

Note that $\text{Hom}_{B_1}(k, N_{G_2} \otimes \lambda) = 0$ unless $\lambda \in pX(T)$.

Suppose $\lambda = p\sigma - \alpha_2$, $\lambda = p\sigma - (\alpha_1 + \alpha_2)$, or $\lambda = p\sigma - 2\alpha_1$ for $\sigma \in X(T)$. Then $E_2^{i,0} = 0$ and

$$\begin{aligned} H^1(B_r, N_{G_2} \otimes \lambda) &\cong E_2^{0,1} = \text{Hom}_{B_r/B_1}(k, H^1(B_1, N_{G_2} \otimes \lambda)) \\ &\cong \text{Hom}_{B_{r-1}}(k, H^1(B_1, N_{G_2} \otimes \lambda)^{(-1)})^{(1)}. \end{aligned}$$

In the first case $H^1(B_1, N_{G_2} \otimes \lambda)^{(-1)} = N_{G_2} \otimes \sigma$ which has socle σ . Hence, to have a non-zero Hom-group, we need $\sigma = p^{r-1}\gamma$ for some $\gamma \in X(T)$. In the last two cases, $H^1(B_1, N_{G_2} \otimes \lambda)^{(-1)} \cong \sigma$ and so again $\sigma = p^{r-1}\gamma$ for $\gamma \in X(T)$.

We summarize: for $r > 1$ and $\lambda = p\sigma - \alpha_2$, $\lambda = p\sigma - (\alpha_1 + \alpha_2)$, or $\lambda = p\sigma - 2\alpha_1$,

$$H^1(B_r, N_{G_2} \otimes \lambda)^{(-r)} \cong \begin{cases} \gamma & \text{if } \lambda = p^r \gamma - \alpha_2, \\ \gamma & \text{if } \lambda = p^r \gamma - (\alpha_1 + \alpha_2), \\ \gamma & \text{if } \lambda = p^r \gamma - 2\alpha_1 \\ 0 & \text{else.} \end{cases} \quad (5.5.2)$$

For all other λ , we have $E^{i,1} = 0$ and so

$$H^1(B_r, N_{G_2} \otimes \lambda) \cong E_2^{1,0} \cong H^1(B_{r-1}, \text{Hom}_{B_1}(k, N_{G_2} \otimes \lambda)^{(-1)})^{(1)}.$$

As noted above, for this to be non-zero, we need $\lambda = p\sigma$ for some $\sigma \in X(T)$, in which case we get $H^1(B_r, N_{G_2} \otimes \lambda) \cong H^1(B_{r-1}, \sigma)^{(1)}$. From [BNP, 2.8], this is zero unless $\sigma = p^{r-1}\gamma - p^l\alpha$ for $\alpha \in \Delta$ and $0 \leq l \leq r-2$. For such σ , the answer is $p^r\gamma$ except in the case that $\alpha = \alpha_2$ and $l = r-2$ in which case the answer is $N_{G_2}^{(r)} \otimes p^r\gamma$. Solving for λ and combining the results with (5.5.1) and (5.5.2) proves the claim. \square

5.6 We now compute $H^2(B_r, \lambda)$ for some special weights.

Lemma *Let $p \geq 3$, $0 \leq l < r$, and $\alpha \in \Delta$.*

(a) *Then*

$$H^2(B_r, -p^l\alpha) \cong \begin{cases} k & \text{if } l > 0, \\ 0 & \text{if } l = 0. \end{cases}$$

(b) *Suppose $p = 3$ and Φ is of type A_2 . Then for $i \in \{1, 2\}$,*

$$H^2(B_r, -p^l\omega_i) \cong \begin{cases} k & \text{if } l > 0, \\ 0 & \text{if } l = 0. \end{cases}$$

Proof. For part (a), consider first the case $r = 1$. From Corollary 5.3, we have $H^2(B_1, -\alpha) = 0$ as desired. Now assume $r > 1$ and consider the case that $l = 0$. We use the LHS spectral sequence

$$E_2^{i,j} = H^i(B_r/B_1, H^j(B_1, -\alpha)) \Rightarrow H^{i+j}(B_r, -\alpha).$$

Since $p \geq 3$ we have $E_2^{i,0} = 0$. Furthermore, from the first case, we have $E_2^{i,2} = 0$. Finally, we look at $E_2^{1,1} = H^1(B_r/B_1, H^1(B_1, -\alpha))$. If $p = 3$ and Φ is of type A_2 we have $H^1(B_1, -\alpha_i) \cong k \oplus (\omega_i - \alpha_i)$ [Jan2, 3.5] and from [BNP, Thm 2.8(B)] that $E_2^{1,1} = 0$. Similarly, it follows, for $p = 3$ and Φ of type G_2 , from [Jan2, 3.7 (b)], [BNP, Thm 2.8(B)], and Lemma 5.5(B) (for $\alpha = \alpha_2$) that $E_2^{1,1} = H^1(B_r/B_1, H^1(B_1, -\alpha)) = 0$. In all other cases $H^1(B_1, -\alpha) = k$ by

[Jan2, 3.5] and $E_2^{1,1}$ vanishes. Hence $E_2^{2,0} = E_2^{1,1} = E_2^{0,2} = 0$ in all cases and $H^2(B_r, -\alpha) = 0$.

Next assume that $l > 0$. We use the LHS spectral sequence

$$E_2^{i,j} = H^i(B_r/B_l, H^j(B_l, k) \otimes -p^l\alpha) \Rightarrow H^{i+j}(B_r, -p^l\alpha).$$

Since $E_2^{i,1} = 0$, we obtain the five-term exact sequence $0 \rightarrow E_2^{2,0} \rightarrow E^2 \rightarrow E_2^{0,2} \rightarrow E_2^{3,0}$. From the case $l = 0$ one concludes that

$$E_2^{2,0} = H^2(B_r/B_l, -p^l\alpha) \cong H^2(B_{r-l}, -\alpha)^{(l)} = 0.$$

Proposition 5.4 and Lemma 5.4 imply that

$$\begin{aligned} E_2^{0,2} &= \text{Hom}_{B_r/B_l}(k, H^2(B_l, k) \otimes -p^l\alpha) \\ &\cong \text{Hom}_{B_r/B_l}(k, H^2(B_1, k)^{(l-1)} \otimes -p^l\alpha) \cong k. \end{aligned}$$

It suffices now to prove that the B -map $E_2^{0,2} \rightarrow E_2^{3,0}$ is zero. Observe that $E_2^{3,0} \cong H^3(B_{r-l}, -\alpha)^{(l)}$. We will show that $\text{Hom}_B(k, H^3(B_m, -\alpha)) = 0$ for $m > 0$, by using the spectral sequence

$$E_2^{i,j} = H^i(B/B_m, H^j(B_m, -\alpha)) \Rightarrow H^{i+j}(B, -\alpha).$$

Clearly $E_2^{i,0} = 0$ and $E_2^{i,2} = 0$ as before. Moreover, by [BNP, 2.8] and some straightforward height considerations in the cases $p = 3$ and Φ of type A_2 or G_2 (cf. [Jan1, II.4.10]) one obtains

$$E_2^{i,1} = H^i(B/B_m, H^1(B_m, -\alpha)) \cong \begin{cases} k & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We conclude that $\text{Hom}_{B/B_m}(k, H^3(B_m, -\alpha)) = E_2^{0,3} \cong E^3 = H^3(B, -\alpha)$. The last term vanishes because of height considerations [Jan1, II.4.10].

For part (b), the argument is analogous. The case $r = 1$ again follows from Corollary 5.3. For $r > 1$ and $l = 0$, we use the spectral sequence

$$E_2^{i,j} = H^i(B_r/B_1, H^j(B_1, -\omega_i)) \Rightarrow H^{i+j}(B_r, -\omega_i).$$

In this case, by [Jan2, 3.5], $H^1(B_1, -\omega_i) = 0$ and so one immediately gets $E_2^{i,1} = 0$ along with $E_2^{i,0} = 0 = E_2^{i,2}$ as before. For the case $l > 0$, one again uses an analogous spectral sequence. Here Lemma 5.4(b) is used and the fact that $H^3(B, -\omega_i) = 0$ follows because ω_i is not in the root lattice [Jan1, II.4.10]. \square

5.7 The preceding calculations can be used to compute $H^2(B_r, \lambda)$ for any r and $\lambda \in X(T)$. For $r = 1$, the results reduce to those given in Theorem 5.3,

although the results here are stated for an arbitrary (not necessarily restricted) weight. One obtains a nice uniform picture except for certain weights when $p = 3$ and Φ is of type A_2 or G_2 .

Theorem *Let $p \geq 3$ and $\lambda \in X(T)$.*

(a) *If $p \neq 3$ or Φ is not of type A_2 or G_2 , then $H^2(B_r, \lambda) \cong$*

$$\left\{ \begin{array}{ll} H^2(B_1, w \cdot 0 + p\gamma)^{(r-1)} & \text{if } \lambda = p^{r-1}(w \cdot 0 + p\gamma) \text{ with } l(w) = 2 \text{ or } 0, \\ \gamma^{(r)} & \text{if } \lambda = p^r\gamma + p^l w \cdot 0, \text{ with } l(w) = 2, \ 0 \leq l < r-1, \\ \gamma^{(r)} & \text{if } \lambda = p^r\gamma - p^l\alpha, \text{ with } 0 < l < r \text{ and } \alpha \in \Delta, \\ \gamma^{(r)} & \text{if } \lambda = p^r\gamma - p^k\beta - p^l\alpha, \text{ with } 0 \leq l < k < r \\ & \text{and } \alpha, \beta \in \Delta, \\ 0 & \text{else.} \end{array} \right.$$

(b) *If $p = 3$ and Φ is of type A_2 , then $H^2(B_r, \lambda) \cong$*

$$\left\{ \begin{array}{ll} (\mathbf{u}^{(r)} \oplus \omega_1^{(r)} \oplus \omega_2^{(r)}) \otimes \gamma^{(r)} & \text{if } \lambda = p^r\gamma, \\ \gamma^{(r)} & \text{if } \lambda = p^r\gamma - p^l w \cdot 0, \text{ with } l(w) = 2, \\ & 0 \leq l < r-1, \\ \gamma^{(r)} & \text{if } \lambda = p^r\gamma - p^l\alpha, \text{ with } 0 < l < r \text{ and } \alpha \in \Delta, \\ \gamma^{(r)} & \text{if } \lambda = p^r\gamma - p^k\beta - p^l\alpha, \text{ with } 0 \leq l < k < r-1 \\ & \text{and } \alpha, \beta \in \Delta, \\ (\gamma + \omega_1)^{(r)} \oplus (\gamma + \omega_2)^{(r)} & \text{if } \lambda = p^r\gamma + p^{r-1}(\omega_1 + \omega_2) - p^l\alpha, \\ & \text{with } 0 \leq l < r-1 \text{ and } \alpha \in \Delta, \\ 0 & \text{else.} \end{array} \right.$$

(c) *If $p = 3$ and Φ is of type G_2 , then $H^2(B_r, \lambda) \cong$*

$$\left\{ \begin{array}{ll} H^2(B_1, w \cdot 0 + p\gamma)^{(r-1)} & \text{if } \lambda = p^{r-1}(p\gamma + w \cdot 0) \text{ with } l(w) = 2 \text{ or } 0, \\ \gamma^{(r)} & \text{if } \lambda = p^r\gamma + p^l w \cdot 0, \text{ with } l(w) = 2, \text{ and} \\ & 0 \leq l < r-1, \\ \gamma^{(r)} & \text{if } \lambda = p^r\gamma - p^l\alpha, \text{ with } 0 < l < r \text{ and } \alpha \in \Delta, \\ \gamma^{(r)} & \text{if } \lambda = p^r\gamma - p^k\beta - p^l\alpha, \text{ with } 0 \leq l < k < r \\ & \text{and } \alpha, \beta \in \Delta, \text{ where } k \neq r-1 \text{ if } \beta = \alpha_2, \\ & \text{and } k \neq l+1 \text{ if } \beta = \alpha_1 \text{ and } \alpha = \alpha_2, \\ \gamma^{(r)} & \text{if } \lambda = p^r\gamma - p^{l+1}(\beta + \alpha_1) - p^l\alpha_2, \text{ with} \\ & 0 \leq l < r-1 \text{ and } \beta \in \Delta, \\ N_{G_2}^{(r)} \otimes \gamma^{(r)} & \text{if } \lambda = p^r\gamma - p^{r-1}\alpha_2 - p^l\alpha, \text{ with } 0 \leq l < r-1, \\ & \text{and } \alpha \in \Delta, \\ 0 & \text{else.} \end{array} \right.$$

Proof. We use induction on r . For $r = 1$, the claim reduces to Theorem 5.3. Suppose $r > 1$. Set $\lambda = \lambda_0 + p\lambda_1$ where $\lambda_0 \in X_1(T)$ and $\lambda_1 \in X(T)$. We use the LHS spectral sequence

$$E_2^{i,j} = H^i(B_r/B_1, H^j(B_1, \lambda_0) \otimes p\lambda_1) \Rightarrow H^{i+j}(B_r, \lambda).$$

Case 1: $\lambda_0 \not\equiv 0$ and $\lambda_0 \not\equiv -\alpha \pmod{pX(T)}$, with $\alpha \in \Delta$.

In this case we have $E_2^{i,0} = 0$ and $E_2^{i,1} = 0$ [Jan2, 3.2]. This implies that

$$H^2(B_r, \lambda) = E^2 \cong E_2^{0,2} = \text{Hom}_{B_r/B_1}(k, H^2(B_1, \lambda_0) \otimes p\lambda_1).$$

By Theorem 5.3(b) this expression is zero unless $\lambda_0 = p\gamma_w + w \cdot 0$ for some $w \in W$ with $l(w) = 2$ and γ_w as given in Lemma 5.1.

Now assume λ_0 is of this form. If we are not dealing with one of the exceptions listed in (i) through (x) in 5.3(c), then the B -module $H^2(B_1, \lambda_0)$ has simple socle of weight $p\gamma_w$. Clearly $E_2^{0,2}$ vanishes unless $p(\gamma_w + \lambda_1) \in p^r X(T)$. This implies that $\lambda = w \cdot 0 + p^r \gamma$ with $l(w) = 2$ and $\gamma \in X(T)$. Moreover, one obtains $H^2(B_r, \lambda) \cong \gamma^{(r)}$ for such weights. For the exceptions (i)-(vii) one has $H^2(B_1, \lambda_0) \cong \gamma_{w_1}^{(1)} \oplus \gamma_{w_2}^{(1)}$, corresponding to the two choices w_1, w_2 . Again, $E^2 \cong E_2^{0,2} = 0$ unless $\lambda = w_i \cdot 0 + p^r \gamma$ for some $\gamma \in X(T)$ and $H^2(B_r, w_i \cdot 0 + p^r \gamma)^{(-r)} \cong \gamma$. Exceptions (viii)-(x) have $\lambda_0 \equiv -\alpha \pmod{pX(T)}$ for some $\alpha \in \Delta$ and are therefore excluded. We summarize:

If $\lambda_0 \not\equiv 0$ and $\lambda_0 \not\equiv -\alpha \pmod{pX(T)}$, with $\alpha \in \Delta$, and $r > 1$ then

$$H^2(B_r, \lambda) \cong \begin{cases} \gamma^{(r)} & \text{if } \lambda = p^r \gamma + w \cdot 0, \text{ with } l(w) = 2, \\ 0 & \text{else.} \end{cases}$$

Note that for $p = 3$, type A_2 and $l(w) = 2$ we have $w \cdot 0 + p^r \gamma = p^r \gamma - p\omega_i$ for an appropriate fundamental weight ω_i .

Case 2: $\lambda_0 \equiv -\alpha \pmod{pX(T)}$, with $\alpha \in \Delta$.

From [Jan2, 3.3] we know that $\lambda_0 = p\omega_\alpha - \alpha$ (unless $p = 3$, Φ is of type G_2 , and $\alpha = \alpha_2$, in which case $\lambda_0 = p(\omega_2 - \omega_1) - \alpha_2$). Since $p > 2$ we have $E_2^{i,0} = 0$. It follows from Corollary 5.3(b) that $E_2^{i,2} = 0$. We conclude that

$$E^2 \cong E_2^{1,1} = H^1(B_r/B_1, H^1(B_1, \lambda_0) \otimes p\lambda_1).$$

For the general argument we exclude two cases, namely $p = 3$ and Φ of type A_2 and $p = 3$, Φ of type G_2 , and $\alpha = \alpha_2$. In all other cases we proceed as follows.

We apply [Jan2, 3.5, 3.7] which yields

$$E_2^{1,1} \cong H^1(B_r/B_1, p(\omega_\alpha + \lambda_1)) \cong H^1(B_{r-1}, \omega_\alpha + \lambda_1)^{(1)}.$$

Now [BNP, 2.8] implies that $E_2^{1,1} = 0$ unless $\omega_\alpha + \lambda_1 = p^{r-1}\gamma - p^{k-1}\beta$ for some simple root β and some $0 < k < r$. Moreover, in this case $H^2(B_r, \lambda) \cong \gamma^{(r)}$ unless Φ is of type G_2 , $\beta = \alpha_2$ and $k = r - 1$. In the latter case, $H^2(B_r, \lambda) \cong N_{G_2}^{(r)} \otimes \gamma^{(r)}$.

Next assume that $p = 3$ and Φ is of type A_2 . Here $p\omega_i - \alpha_i = \omega_1 + \omega_2$ and [Jan2, 3.5 (a)] yields $E_2^{1,1} \cong H^1(B_r/B_1, p(\omega_1 \oplus \omega_2) \otimes p\lambda_1) \cong H^1(B_{r-1}, \omega_1 + \lambda_1)^{(1)} \oplus H^1(B_{r-1}, \omega_2 + \lambda_1)^{(1)}$. As before we obtain from [BNP, 2.8] that the cohomology vanishes unless $\lambda = p^r\gamma - p^k\alpha_j - \alpha_i$, where $i, j \in \{1, 2\}$. Moreover, $H^2(B_r, \lambda)^{(-r)} \cong \gamma$, unless $k = r - 1$ in which case $H^2(B_r, \lambda)^{(-r)} \cong \gamma \oplus (\gamma + (-1)^j(\omega_1 - \omega_2))$. Adding $p^r\omega_j$ to λ results in the more symmetric statement $H^2(B_r, p^r\gamma + p^{r-1}(\omega_1 + \omega_2) - \alpha_i)^{(-r)} \cong (\gamma + \omega_1) \oplus (\gamma + \omega_2)$.

Finally assume that $p = 3$, Φ is of type G_2 , and $\lambda_0 \equiv -\alpha_2 \pmod{pX(T)}$. Define $\gamma \in X(T)$ via $\lambda = p\gamma - \alpha_2$. Then it follows from [Jan2, 3.7] that $H^1(B_1, \lambda) \cong N_{G_2}^{(1)} \otimes \gamma^{(1)}$ and so $H^2(B_r, \lambda) \cong H^1(B_{r-1}, N_{G_2} \otimes \gamma)^{(1)}$. We apply Lemma 5.5(B) and summarize:

If $\lambda_0 \equiv -\alpha \pmod{pX(T)}$, with $\alpha \in \Delta$ and $r > 1$, then

(a) If $p \neq 3$ or Φ is not of type A_2 or G_2 then

$$H^2(B_r, \lambda)^{(-r)} \cong \begin{cases} \gamma & \text{if } \lambda = p^r\gamma - p^k\beta - \alpha, \text{ with } 0 < k < r, \text{ and } \alpha, \beta \in \Delta, \\ 0 & \text{else.} \end{cases}$$

(b) If $p = 3$ and Φ is of type A_2 , then $H^2(B_r, \lambda)^{(-r)} \cong$

$$\begin{cases} \gamma & \text{if } \lambda = p^r\gamma - p^k\beta - \alpha, \text{ with } 0 < k < r - 1 \\ & \text{and } \alpha, \beta \in \Delta, \\ (\gamma + \omega_1) \oplus (\gamma + \omega_2) & \text{if } \lambda = p^r\gamma + p^{r-1}(\omega_1 + \omega_2) - \alpha, \text{ with } \alpha \in \Delta, \\ 0 & \text{else.} \end{cases}$$

(c) If $p = 3$ and Φ is of type G_2 , then $H^2(B_r, \lambda)^{(-r)} \cong$

$$\begin{cases} \gamma & \text{if } \lambda = p^r\gamma - p^k\beta - \alpha, \text{ with } 0 < k < r, \text{ and } \alpha, \beta \in \Delta, \\ & \text{where } k \neq r - 1 \text{ if } \beta = \alpha_2 \text{ and } k \neq 1 \text{ if } \beta = \alpha_1 \text{ and } \alpha = \alpha_2, \\ \gamma & \text{if } \lambda = p^r\gamma - p(\beta + \alpha_1) - \alpha_2, \beta \in \Delta, \\ N_{G_2} \otimes \gamma & \text{if } \lambda = p^r\gamma - p^{r-1}\alpha_2 - \alpha, \text{ with } 0 \leq l < r - 1, \text{ and } \alpha \in \Delta, \\ 0 & \text{else.} \end{cases}$$

Case 3: $\lambda_0 = 0$.

Since p is odd we have $E_2^{i,1} = 0$ for all i by [And]. From Lemma 5.4 one obtains that $E_2^{0,2} = 0$ unless $\lambda = p^r \gamma - p\alpha$, with $\alpha \in \Delta$ or $p = 3$, $\Phi = A_2$, and $\lambda = p^r \gamma - p\omega_i = p^r \gamma + w \cdot 0$, where $l(w) = 2$. By Lemma 5.5 it follows that in these cases $H^2(B_r, \lambda) \cong p^r \gamma$, as claimed. Next we assume that $E_2^{0,2} = 0$. This implies that

$$E^2 \cong E_2^{2,0} \cong H^2(B_r/B_1, p\lambda_1) \cong H^2(B_{r-1}, \lambda_1)^{(1)}.$$

The assertion follows via the induction hypotheses and the previous cases. \square

5.8 B -cohomology: Theorem 5.7 can be used to compute $H^2(B, \lambda)$ for all $\lambda \in X(T)$. Partial computations are given in work of O'Halloran [OHal1, OHal2] and Andersen [And]. To compute $H^2(B, \lambda)$, we use [CPS, Cor. 7.2] which says that $H^2(B, \lambda) \cong \varprojlim H^2(B_r, \lambda)$.

Assume that $\lambda \in X(T)$ with $H^2(B, \lambda) \neq 0$. Clearly $\lambda \neq 0$. From the above isomorphism, we can choose $s > 0$ such that

(i) the natural map $H^2(B, \lambda) \rightarrow H^2(B_r, \lambda)$ is nonzero for all $r \geq s$.

By choosing a possibly larger s , we can further assume that

(ii) $|\langle \lambda, \alpha^\vee \rangle| < p^{s-1}$ for all $\alpha \in \Delta$.

From Theorem 5.7 and condition (ii), one concludes that $H^2(B_r, \lambda)$ is one-dimensional for all $r \geq s$. Since $H^2(B, \lambda)$ has trivial B -action, condition (i) then implies that $H^2(B_r, \lambda) \cong k$ for all $r \geq s$.

On the other hand, if there exists an integer s such that $H^2(B_r, \lambda) \cong k$ for all $r \geq s$, then $H^2(B, \lambda) \cong \varprojlim H^2(B_r, \lambda) \cong k$. Therefore Theorem 5.7 yields:

Theorem *Let $p \geq 3$ and $\lambda \in X(T)$.*

(a) *Suppose $p > 3$ or Φ is not of type G_2 . Then*

$$H^2(B, \lambda) \cong \begin{cases} k & \text{if } \lambda = p^l w \cdot 0, \text{ with } 0 \leq l, l(w) = 2, \\ k & \text{if } \lambda = -p^l \alpha, \text{ with } 0 < l \text{ and } \alpha \in \Delta, \\ k & \text{if } \lambda = -p^k \beta - p^l \alpha, \text{ with } 0 \leq l < k \text{ and } \alpha, \beta \in \Delta, \\ 0 & \text{else.} \end{cases}$$

(b) Suppose $p = 3$ and Φ is of type G_2 . Then

$$H^2(B, \lambda) \cong \begin{cases} k & \text{if } \lambda = p^l w \cdot 0, \text{ with } 0 \leq l, l(w) = 2, \\ k & \text{if } \lambda = -p^l \alpha, \text{ with } 0 < l \text{ and } \alpha \in \Delta, \\ k & \text{if } \lambda = -p^k \beta - p^l \alpha, \text{ with } 0 \leq l < k \text{ and } \alpha, \beta \in \Delta, \\ & \text{where } k \neq l + 1 \text{ if } \beta = \alpha_1 \text{ and } \alpha = \alpha_2, \\ k & \text{if } \lambda = -p^{l+1}(\beta + \alpha_1) - p^l \alpha_2, \text{ with } 0 \leq l \text{ and } \beta \in \Delta, \\ 0 & \text{else.} \end{cases}$$

6 G_r -cohomology

6.1 The computation of B_r -cohomology can now be used to determine the G_r -cohomology of induced modules $H^0(\lambda)$ for $\lambda \in X(T)_+$. In degree one cohomology, one has the isomorphism [Jan1, II.12.2]

$$H^1(G_r, H^0(\lambda))^{(-r)} \simeq \text{ind}_B^G(H^1(B_r, \lambda))^{(-r)}$$

for any $\lambda \in X(T)_+$. This isomorphism holds independently of the prime and was used in [BNP] to give an explicit description of $H^1(G_r, H^0(\lambda))$ for all primes. The following theorem uses the calculations done by the authors in [BNP] to show that this isomorphism can be extended further to higher B_r and G_r -cohomology.

Theorem *Let $\lambda \in X(T)_+$ and p be an arbitrary prime. Then*

$$H^2(G_r, H^0(\lambda))^{(-r)} \simeq \text{ind}_B^G(H^2(B_r, \lambda))^{(-r)}.$$

Proof. Consider the spectral sequence (cf. [Jan1, II.12.2])

$$E_2^{i,j} = R^i \text{ind}_B^G \left(H^j(B_r, \lambda)^{(-r)} \right) \Rightarrow H^{i+j}(G_r, \text{ind}_B^G \lambda)^{(-r)} = H^{i+j}(G_r, H^0(\lambda))^{(-r)}.$$

Since $\lambda \in X(T)_+$, we can decompose $\lambda = \lambda_0 + p^r \lambda_1$ where $\lambda_0 \in X_r(T)$ and $\lambda_1 \in X(T)_+$. Now $\text{Hom}_{B_r}(k, \lambda) \cong \text{Hom}_{B_r}(k, \lambda_0) \otimes p^r \lambda_1$ which is zero for $\lambda_0 \neq 0$ and equal to $p^r \lambda_1$ for $\lambda_0 = 0$. In either case we have $E_2^{i,0} = 0$ for all $i > 0$ by using Kempf's vanishing theorem.

Now consider $H^1(B_r, \lambda)^{(-r)}$. Let μ be a B -composition factor. From [BNP, Thm. 2.8 (A)-(C)], either $\mu \in X(T)_+$ or $\langle \mu, \alpha^\vee \rangle = -1$ for some $\alpha \in \Delta$. According to Kempf's vanishing theorem or [Jan1, II.5.4], for such a μ , one has $R^i \text{ind}_B^G \mu = 0$ for $i > 0$. Therefore, by [BNP, 3.3] (or see the proof of Lemma 6.2 below), $E_2^{i,1} = 0$ for $i > 0$ and so $E_2 = E_2^{0,2}$. \square

6.2 In particular, for $r = 1$, we have

$$H^2(G_1, H^0(\lambda))^{(-1)} \cong \text{ind}_B^G(H^2(B_1, \lambda))^{(-1)}.$$

We immediately obtain the following from Theorem 5.3 - compare to Theorem 1.1.

Theorem *Let $p \geq 3$ and $\lambda \in X(T)_+$.*

(a) *If $\lambda = p\gamma$, then $H^2(G_1, H^0(\lambda)) \cong \text{ind}_B^G(\mathbf{u}^* \otimes \gamma)^{(1)}$ except in the following cases:*

(i) $p = 3$, Φ is of type A_2 , where

$$H^2(G_1, H^0(\lambda)) \cong \text{ind}_B^G(\mathbf{u}^* \otimes \gamma)^{(1)} \oplus H^0(\omega_1 + \gamma)^{(1)} \oplus H^0(\omega_2 + \gamma)^{(1)}.$$

(ii) $p = 3$, Φ is of type G_2 , where

$$H^2(G_1, H^0(\lambda)) \cong \text{ind}_B^G(H^1(B_1, k))^{(-1)} \otimes \gamma)^{(1)}.$$

(b) *If $\lambda \notin pX(T)$ and $H^2(G_1, H^0(\lambda)) \neq 0$, then $\lambda = w \cdot 0 + p\gamma$ for some $w \in W$ with $l(w) = 2$ and $\gamma \in X(T)$.*

(c) *If $p \geq 5$ and $\lambda = w \cdot 0 + p\gamma$ for some $w \in W$ with $l(w) = 2$ and $\gamma \in X(T)$, then*

$$H^2(G_1, H^0(\lambda)) \cong H^0(\gamma)^{(1)},$$

except in the following case:

$p = 5$, Φ is of type A_4 , and $w \in \{s_{\alpha_2}s_{\alpha_1}, s_{\alpha_3}s_{\alpha_4}\}$, where

$$H^2(G_1, H^0(\lambda)) \cong H^0(\gamma)^{(1)} \oplus H^0(\gamma - \omega_2 + \omega_3)^{(1)}.$$

For $p = 3$ and $\lambda = w \cdot 0 + p\gamma \notin pX(T)$, we usually have $H^2(G_1, H^0(\lambda)) \cong H^0(\gamma)^{(1)}$ with exceptions due to vanishing or doubling of cohomology coming from cases (i)-(x) of Theorem 5.3. The computations are straightforward and left to the interested reader. Cases (xi)-(xv) of Theorem 5.3 lead to the computation of induced modules for certain two-dimensional indecomposable B -modules. We discuss how to deal with these. Consider the following modules:

- Φ is of type B_n with $n \geq 3$: N_{B_n} has factors α_n and k corresponding to $w = s_{\alpha_{n-1}}s_{\alpha_{n-2}}$.
- Φ is of type C_n with $n \geq 3$: N_{C_n} has factors α_{n-1} and k corresponding to $w = s_{\alpha_{n-2}}s_{\alpha_n}$.
- Φ is of type F_4 : N_{F_4} has factors α_3 and k corresponding to $w = s_{\alpha_2}s_{\alpha_1}$ or $w = s_{\alpha_2}s_{\alpha_4}$.
- Φ is of type G_2 : N_{G_2} has factors α_1 and k corresponding to $w = s_{\alpha_2}s_{\alpha_1}$.

If $\lambda = w \cdot 0 + p\gamma \in X(T)_+$ for the above $w \in W$, then $H^2(G_1, H^0(\lambda)) \cong \text{ind}_B^G(N_{X_n} \otimes \gamma)^{(1)}$. The module $\text{ind}_B^G(N_{X_n} \otimes \gamma)$ always admits a good filtration as given by the following lemma. All factors are listed from top to bottom.

Lemma *Let $p = 3$ and N_{X_n} be a module as above with corresponding $w \in W$. Suppose $\gamma \in X(T)$ with $w \cdot 0 + p\gamma \in X(T)_+$.*

- (a) Φ is of type B_n with $n \geq 3$: Then $\langle \gamma, \alpha_i^\vee \rangle \geq 0$ for $1 \leq i \leq n-2$, $\langle \gamma, \alpha_{n-1}^\vee \rangle \geq 1$, and $\langle \gamma, \alpha_n^\vee \rangle \geq -1$. Further,
(i) If $\langle \gamma, \alpha_n^\vee \rangle = -1$, then $\text{ind}_B^G(N_{B_n} \otimes \gamma) \cong H^0(\alpha_n + \gamma)$.
(ii) If $\langle \gamma, \alpha_n^\vee \rangle \geq 0$, then $\text{ind}_B^G(N_{B_n} \otimes \gamma)$ has a filtration with factors $H^0(\alpha_n + \gamma)$ and $H^0(\gamma)$.
- (b) Φ is of type C_n with $n \geq 3$: Then $\langle \gamma, \alpha_i^\vee \rangle \geq 0$ for $1 \leq i \leq n-3$, $\langle \gamma, \alpha_i^\vee \rangle \geq 1$ for $i \in \{n-2, n\}$, and $\langle \gamma, \alpha_{n-1}^\vee \rangle \geq -1$. Further,
(i) If $\langle \gamma, \alpha_{n-1}^\vee \rangle = -1$, then $\text{ind}_B^G(N_{C_n} \otimes \gamma) \cong H^0(\alpha_{n-1} + \gamma)$.
(ii) If $\langle \gamma, \alpha_{n-1}^\vee \rangle \geq 0$, then $\text{ind}_B^G(N_{C_n} \otimes \gamma)$ has a filtration with factors $H^0(\alpha_{n-1} + \gamma)$ and $H^0(\gamma)$.
- (c) Φ is of type F_4 and $w = s_{\alpha_2} s_{\alpha_1}$: Then $\langle \gamma, \alpha_1^\vee \rangle \geq 0$, $\langle \gamma, \alpha_2^\vee \rangle \geq 1$, $\langle \gamma, \alpha_3^\vee \rangle \geq -1$, and $\langle \gamma, \alpha_4^\vee \rangle \geq 0$. Further,
(i) If $\langle \gamma, \alpha_3^\vee \rangle = -1$ and $\langle \gamma, \alpha_4^\vee \rangle = 0$, then $\text{ind}_B^G(N_{F_4} \otimes \gamma) = 0$.
(ii) If $\langle \gamma, \alpha_3^\vee \rangle = -1$ and $\langle \gamma, \alpha_4^\vee \rangle \geq 1$, then $\text{ind}_B^G(N_{F_4} \otimes \gamma) \cong H^0(\alpha_3 + \gamma)$.
(iii) If $\langle \gamma, \alpha_3^\vee \rangle \geq 0$ and $\langle \gamma, \alpha_4^\vee \rangle = 0$, then $\text{ind}_B^G(N_{F_4} \otimes \gamma) \cong H^0(\gamma)$.
(iv) If $\langle \gamma, \alpha_3^\vee \rangle \geq 0$ and $\langle \gamma, \alpha_4^\vee \rangle \geq 1$, then $\text{ind}_B^G(N_{F_4} \otimes \gamma)$ has a filtration with factors $H^0(\alpha_3 + \gamma)$ and $H^0(\gamma)$.
- (d) Φ is of type F_4 and $w = s_{\alpha_2} s_{\alpha_4}$: Then $\langle \gamma, \alpha_1^\vee \rangle \geq 0$, $\langle \gamma, \alpha_2^\vee \rangle \geq 1$, $\langle \gamma, \alpha_3^\vee \rangle \geq -1$, and $\langle \gamma, \alpha_4^\vee \rangle \geq 1$. Further,
(i) If $\langle \gamma, \alpha_3^\vee \rangle = -1$, then $\text{ind}_B^G(N_{F_4} \otimes \gamma) \cong H^0(\alpha_3 + \gamma)$.
(ii) If $\langle \gamma, \alpha_3^\vee \rangle \geq 0$, then $\text{ind}_B^G(M_{F_4} \otimes \gamma)$ has a filtration with factors $H^0(\alpha_3 + \gamma)$ and $H^0(\gamma)$.
- (e) Φ is of type G_2 : Then $\langle \gamma, \alpha_1^\vee \rangle \geq -1$ and $\langle \gamma, \alpha_2^\vee \rangle \geq 1$. Further,
(i) If $\langle \gamma, \alpha_1^\vee \rangle = -1$, then $\text{ind}_B^G(N_{G_2} \otimes \gamma) \cong H^0(\alpha_1 + \gamma)$.
(ii) If $\langle \gamma, \alpha_1^\vee \rangle \geq 0$, then $\text{ind}_B^G(N_{G_2} \otimes \gamma)$ has a filtration with factors $H^0(\alpha_1 + \gamma)$ and $H^0(\gamma)$.

Proof. The claims about γ follow by direct computation. For example, in part (a), we have $w \cdot 0 = \omega_{n-3} - 3\omega_{n-1} + 4\omega_n$. Write $\gamma = \sum_{i=1}^n c_i \omega_i$. To have $3\gamma + w \cdot 0$ being dominant, we evidently need $c_i \geq 0$ for $1 \leq i \leq n-2$, $c_{n-1} \geq 1$, and $c_n \geq -1$ as claimed.

For the structure of the induced modules, we argue as in [BNP, Proposition 3.4] using [BNP, Lemma 3.3]. To summarize, if N is an indecomposable B -module with factors σ_1, σ_2 and $R^i \text{ind}_B^G \sigma_j = 0$ for $i \geq 1$ and each j , then $\text{ind}_B^G N$ has a filtration with factors $H^0(\sigma_1), H^0(\sigma_2)$ where either (or both) factor is omitted if σ_j is not dominant.

We again go through part (a) and leave the others to the interested reader. Here $N_{B_n} \otimes \gamma$ has factors $\alpha_n + \gamma$ and γ . Notice that the top weight is dominant and so $R^i \text{ind}_B^G(\alpha_n + \gamma) = 0$ for all $i > 0$ (cf. [Jan1, II.4.5]). The bottom weight, which is simply γ , need not be dominant. If $\langle \gamma, \alpha_n \rangle = -1$, then γ is

not dominant, but $R^i \text{ind}_B^G \gamma = 0$ for all $i \geq 0$ (cf. [Jan1, II.5.4(a)]) and so part (i) follows. On the other hand, if $\langle \gamma, \alpha_n \rangle \geq 0$, then γ is dominant and part (ii) follows. \square

6.3 General Case: For $r > 1$, we can get the results from Theorem 5.7.

Theorem *Let $p \geq 3$, $r > 1$, and $\lambda \in X(T)_+$.*

(a) *If $p \neq 3$ or Φ is not of type A_2 or G_2 , then $H^2(G_r, H^0(\lambda)) \cong$*

$$\left\{ \begin{array}{ll} \text{ind}_B^G(H^2(B_1, w \cdot 0 + p\gamma))^{(r-1)} & \text{if } \lambda = p^{r-1}(w \cdot 0 + p\gamma) \text{ with } l(w) = 2 \text{ or } 0, \\ H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma + p^l w \cdot 0, \text{ with } l(w) = 2, \\ & 0 \leq l < r - 1, \\ H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^l\alpha, \text{ with } 0 < l < r \text{ and } \alpha \in \Delta, \\ H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^k\beta - p^l\alpha, \text{ with } 0 \leq l < k < r \\ & \text{and } \alpha, \beta \in \Delta, \\ 0 & \text{else.} \end{array} \right.$$

(b) *If $p = 3$ and Φ is of type A_2 , then $H^2(G_r, H^0(\lambda)) \cong$*

$$\left\{ \begin{array}{ll} \text{ind}_B^G(\mathbf{u}^* \otimes \gamma)^{(r)} \oplus H^0(\omega_1 \otimes \gamma)^{(r)} \oplus & H^0(\omega_2 \otimes \gamma)^{(r)} \\ & \text{if } \lambda = p^r\gamma, \\ H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^l w \cdot 0, \text{ with } l(w) = 2, \\ & 0 \leq l < r - 1, \\ H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^l\alpha, \text{ with } 0 < l < r \\ & \text{and } \alpha \in \Delta, \\ H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^k\beta - p^l\alpha, \text{ with} \\ & 0 \leq l < k < r - 1 \text{ and } \alpha, \beta \in \Delta, \\ H^0(\gamma + \omega_1)^{(r)} \oplus H^0(\gamma + \omega_2)^{(r)} & \text{if } \lambda = p^r\gamma + p^{r-1}(\omega_1 + \omega_2) - p^l\alpha, \\ & \text{with } 0 \leq l < r - 1 \text{ and } \alpha \in \Delta, \\ 0 & \text{else.} \end{array} \right.$$

(c) If $p = 3$ and Φ is of type G_2 , then $H^2(G_r, H^0(\lambda)) \cong$

$$\left\{ \begin{array}{ll} \text{ind}_B^G(H^2(B_1, w \cdot 0 + p\gamma))^{(r-1)} & \text{if } \lambda = p^{r-1}(p\gamma + w \cdot 0) \text{ with } l(w) = 2 \text{ or } 0, \\ H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma + p^l w \cdot 0, \text{ with } l(w) = 2, \\ & \text{and } 0 \leq l < r - 1, \\ H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^l\alpha, \text{ with } 0 < l < r \text{ and } \alpha \in \Delta, \\ H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^k\beta - p^l\alpha, \text{ with } 0 \leq l < k < r \\ & \text{and } \alpha, \beta \in \Delta, \\ & \text{where } k \neq r - 1 \text{ if } \beta = \alpha_2, \\ & \text{and } k \neq l + 1 \text{ if } \beta = \alpha_1 \text{ and } \alpha = \alpha_2, \\ H^0(\gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^{l+1}(\beta + \alpha_1) - p^l\alpha_2, \\ & \text{with } 0 \leq l < r - 1 \text{ and } \beta \in \Delta, \\ \text{ind}_B^G(N_{G_2} \otimes \gamma)^{(r)} & \text{if } \lambda = p^r\gamma - p^{r-1}\alpha_2 - p^l\alpha, \\ & \text{with } 0 \leq l < r - 1 \text{ and } \alpha \in \Delta, \\ 0 & \text{else.} \end{array} \right.$$

For $r > 1$, $p = 3$, and Φ of type G_2 , the last non-trivial case gives rise to an additional situation where one has to induce a two-dimensional indecomposable module. Suppose $\lambda = p^r\gamma - p^{r-1}\alpha_2 - p^l\alpha$ with $0 \leq l < r - 1$ and $\alpha \in \Delta$. In order for λ to be dominant, we must have $\langle \gamma, \alpha_1^\vee \rangle \geq 0$ and $\langle \gamma, \alpha_2^\vee \rangle \geq 1$. Hence, as in Lemma 6.2, $\text{ind}_B^G(N_{G_2} \otimes \gamma)$ has a filtration with factors $H^0(\alpha_1 + \gamma)$ and $H^0(\gamma)$.

Donkin [Don, p. 79] conjectured that if V is a rational G -module with good filtration then $H^m(G_r, V)^{(-r)}$ has a good filtration for every $m \geq 0$. van der Kallen [vdK] proved Donkin's conjecture for rank one groups. In the same paper he constructed a counterexample to Donkin's conjecture for higher rank. The results above demonstrate that $H^2(G_r, V)^{(-r)}$ indeed has a good filtration for $p \geq 3$, $V = H^0(\lambda)$, λ a dominant weight, and arbitrary rank. In [BNP], it was verified that Donkin's conjecture holds for $H^1(G_r, H^0(\lambda))$ for all primes and all ranks. An interesting question would be to see if $H^m(G_r, H^0(\lambda))$ admits a good filtration in higher ranks for all $m \geq 0$ and all primes.

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