COHOMOLOGY FOR QUANTUM GROUPS VIA THE GEOMETRY OF THE NULLCONE

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ABSTRACT. Let ζ be a complex ℓth root of unity for an odd integer $\ell > 1$. For any complex simple Lie algebra $\mathfrak{g}$, let $u_\zeta = u_\zeta(\mathfrak{g})$ be the associated "small" quantum enveloping algebra. This algebra is a finite dimensional Hopf algebra which can be realized as a subalgebra of the Lusztig (divided power) quantum enveloping algebra U_ζ and as a quotient algebra of the De Concini–Kac quantum enveloping algebra $\mathcal{U}_\zeta$. It plays an important role in the representation theories of both U_ζ and $\mathcal{U}_\zeta$ in a way analogous to that played by the restricted enveloping algebra u of a reductive group G in positive characteristic p with respect to its distribution and enveloping algebras. In general, little is known about the representation theory of quantum groups (resp., algebraic groups) when l (resp., p) is smaller than the Coxeter number h of the underlying root system. For example, Lusztig's Conjecture concerning the irreducible modules can only be formulated when $p \geq h$. The main result in this paper provides a surprisingly uniform answer for the cohomology algebra $\mathrm{H}^\bullet(u_\zeta, \mathbb{C})$ of the small quantum group. When $\ell > h$, this cohomology algebra has been calculated by Ginzburg and Kumar [GK]. Our result requires powerful tools from complex geometry and a detailed knowledge of the geometry of the nullcone of $\mathfrak{g}$. In this way, the methods point out difficulties present in obtaining similar results for the restricted enveloping algebra u in small characteristics, though they do provide some clarification of known results there also. Finally, we establish that if M is a finite dimensional u_ζ-module, then $\mathrm{H}^\bullet(u_\zeta, M)$ is a finitely generated $\mathrm{H}^\bullet(u_\zeta, \mathbb{C})$-module, and we obtain new results on the theory of support varieties for u_ζ.

1. Introduction

1.1. Let $\mathfrak{g}_F$ be a finite dimensional, restricted Lie algebra (as defined by Jacobson) over an algebraically closed field F of positive characteristic p, with restriction map $x \mapsto x^{[p]}$, $x \in \mathfrak{g}_F$. The restricted enveloping algebra $u := u(\mathfrak{g}_F)$ of $\mathfrak{g}_F$ is a finite dimensional cocommutative Hopf algebra. In general, the cohomology algebra $A := \mathrm{H}^\bullet(u, F)$ is difficult to compute. However, Suslin-Friedlander-Bendel [SFB1, SFB2] proved that the (algebraic) scheme $\operatorname{Spec} A$ is homeomorphic to the closed subvariety $\mathcal{N}_1(\mathfrak{g}_F) := \{x \in \mathfrak{g}_F : \ x^{[p]} = 0\}$. We call $\mathcal{N}_1(\mathfrak{g}_F)$ the restricted nullcone of $\mathfrak{g}_F$; it is a closed subvariety of the full nullcone $\mathcal{N}(\mathfrak{g}_F)$ which consists of all $[p]$-nilpotent elements in $\mathfrak{g}_F$.

When $\mathfrak{g}_F$ is the Lie algebra of a reductive algebraic group G over F, the above results can be considerably sharpened. For example, if $p > h$ (the Coxeter number of G), then $\mathrm{H}^{2\bullet}(u, F) \cong F[\mathcal{N}_1(\mathfrak{g}_F)]$, the coordinate algebra of $\mathcal{N}_1(\mathfrak{g}_F)$ (cf. Friedlander-Parshall [FP2]

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and Andersen-Jantzen [AJ]). In fact, the condition $p \geq h$ implies that $\mathcal{N}_1(\mathfrak{g}_F) = \mathcal{N}(\mathfrak{g}_F)$. However, when $p \leq h$, there is no known calculation of $\mathrm{H}^\bullet(u, F)$ (apart from some small rank cases). For all primes, $\mathcal{N}_1(\mathfrak{g}_F)$ is an irreducible variety [NPV, (6.3.1) Cor.], [UGA2, Thm. 4.2] and is the closure of a G-orbit. These orbits have been determined in [CLNP], [UGA2, Thm. 4.2].

Now let $\mathfrak{g} = \mathfrak{g}_\mathbb{C}$ be a complex simple Lie algebra, and let $U_\zeta = U_\zeta(\mathfrak{g})$ be the quantum enveloping algebra (Lusztig form) associated to $\mathfrak{g}$ at a primitive lth root of unity $\zeta \in \mathbb{C}$. We regard U_ζ as an algebra over $\mathbb{C}$ obtained by base change from the quantum enveloping algebra over the cyclotomic field $\mathbb{Q}(\zeta)$. Here $l > 1$ is an odd integer, not divisible by 3 when $\mathfrak{g}$ has type G_2. The representation theory of U_ζ, when $l = p$, models or approximates the representation theory of G. For example, part of the "Lusztig program" to determine the characters of the irreducible G-modules amounts to showing, when $l = p \geq h$, that the characters of the irreducible G-modules having high weights in the so-called Jantzen region coincide with the characters of the analogous irreducible modules for U_ζ. This result has been proved by Andersen-Jantzen-Soergel [AJS] for large p where an effective lower bound is unknown.

For all l, Lusztig [L] has also introduced an analog, denoted $u_\zeta := u_\zeta(\mathfrak{g})$, of the restricted enveloping algebra u. Like u, u_ζ is a finite dimensional Hopf algebra, though it is in general not cocommutative. It plays a role in the representation theory of U_ζ much like that played by u in the representation theory of G. In particular, for $l > h$, Ginzburg-Kumar [GK] have calculated the cohomology algebra $\mathrm{H}^\bullet(u_\zeta, \mathbb{C})$. By exact analogy with the cohomology of u, they prove that $\mathrm{H}^{2\bullet}(u_\zeta, \mathbb{C}) \cong \mathbb{C}[\mathcal{N}(\mathfrak{g})]$, the coordinate algebra of the nullcone $\mathcal{N}(\mathfrak{g})$ consisting of nilpotent elements in $\mathfrak{g}$. Recently, Arkhipov-Bezrukavnikov-Ginzburg [ABG, §1.4], taking [GK] as a method to pass from the representation theory of quantum groups to the geometry of the nullcone, provide a proof of Lusztig's character formula for U_ζ when $l > h$. These important connections have made the small quantum group u_ζ an object of significant interest (cf. [ABBGM], [AG], [Be], [Lac1, Lac2]).

This paper presents new results on the cohomology of u_ζ. In particular, we compute the cohomology algebra $\mathrm{H}^\bullet(u_\zeta, \mathbb{C})$ in the remaining cases when $l \leq h$. Our results are explicitly described in Section 1.3 below. We prove in Sections 2–5 that $\mathrm{H}^{2\bullet+1}(u_\zeta, \mathbb{C}) = 0$, while $\mathrm{H}^{2\bullet}(u_\zeta, \mathbb{C})$ is isomorphic to, in most cases, the coordinate algebra of an explicitly described closed subvariety $\mathcal{N}(\Phi_0)$ of $\mathcal{N}(\mathfrak{g})$ (constructed in a similar way to the variety $\mathcal{N}_1(\mathfrak{g}_F)$ discussed above). In Section 2 we rigorously develop and present new results on the cohomology theory of parabolic subalgebras for quantum groups. The application of powerful tools from complex geometry represent at least one advantage that the quantum enveloping algebra situation has over that in positive characteristic. In Sections 3 and 5, we demonstrate how the Grauert-Riemenschneider theorem and the normality of the orbit closures which equal $\mathcal{N}(\Phi_0)$ play a vital role in carrying out our cohomology calculations.

In Section 6, we prove that $R := \mathrm{H}^\bullet(u_\zeta, \mathbb{C})$ is a finitely generated $\mathbb{C}$-algebra and if M is a finite dimensional u_ζ-module, then $\mathrm{H}^\bullet(u_\zeta, M)$ is finitely generated over R. Section 7 adapts some of our methods to the cohomology algebra $\mathrm{H}^\bullet(u, F)$ in positive characteristic, making some ad hoc computations in [AJ] more transparent. In particular, we can identify key vanishing results, known over $\mathbb{C}$, and as yet unproved in positive characteristic, which would be sufficient to extend the calculations to positive characteristic p.

Finally, by building on results in Section 6, we define support varieties in the quantum setting in Section 8 and exhibit some new calculations on support varieties. The theory of support varieties for the restricted Lie algebra $\mathfrak{g}_F$ attached to a reductive group G in positive characteristic provides evidence for beautiful connections between the representation theory of G and the structure and geometry of the restricted nullcone $\mathcal{N}_1(\mathfrak{g}_F)$ ([NPV],[CLNP], [UGA1], [UGA2]). The results of this paper strongly reinforce this expectation (as do those in [ABG] mentioned above).

1.2. Notation. Let Φ be a finite, irreducible root system. Fix a set $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ of simple roots (labelled in the standard way [Bo, Appendix]), and let Φ^+ (resp. Φ^-) be the corresponding set of positive (resp. negative) roots. Write $Q = Q(\Phi) := \mathbb{Z}\Phi$ (the root lattice) and $Q^+ = Q^+(\Phi) = \mathbb{N}\Phi^+$. The set Φ spans a real vector space $\mathbb{E}$ with positive definite inner product $\langle u, v\rangle$, $u, v \in \mathbb{E}$, adjusted so that $\langle \alpha, \alpha\rangle = 2$ if $\alpha \in \Phi$ is a short root. Thus, if $\alpha \in \Phi$ is a long root, then $\langle \alpha, \alpha\rangle = 4$ (resp. 6) when Φ has type B_n, C_n, F_4 (resp. G_2). For $\alpha \in \Phi$, put $d_\alpha := \frac{\langle \alpha,\alpha\rangle}{2} \in \{1, 2, 3\}$.

For $\alpha \in \Phi$, write $\alpha^\vee = \frac{2}{\langle \alpha,\alpha\rangle}\alpha$; thus, $\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\}$ is the coroot system defined by Φ. For $1 \leq i, j \leq n$, put $c_{i,j} = \langle \alpha_j, \alpha_i^\vee\rangle \in \mathbb{N}$. Then $C := [c_{i,j}]$ is the Cartan matrix of Φ and DC is a symmetric matrix, setting $D = \mathrm{diag}[d_{\alpha_1}, \cdots, d_{\alpha_n}]$. Let $X_+ \subset \mathbb{E}$ be the set of dominant weights, consisting of all $\varpi \in \mathbb{E}$ satisfying $\langle \varpi, \alpha_i^\vee\rangle \in \mathbb{N}$, $i = 1, \cdots, n$. Define the fundamental dominant weights $\varpi_1, \cdots, \varpi_n$ by $\langle \varpi_i, \alpha_j^\vee\rangle = \delta_{i,j}$, so $X_+ = \mathbb{N}\varpi_1 \oplus \cdots \oplus \mathbb{N}\varpi_n$. Put $X = \mathbb{Z}\varpi_1 \oplus \cdots \oplus \mathbb{Z}\varpi_n$ (the weight lattice). The group X is partially ordered: $\lambda \geq \mu \iff \lambda - \mu \in Q^+$. For convenience, we will occasionally use the notation $\varpi_0 := 0$. The Weyl group W is the finite group of orthogonal transformations of $\mathbb{E}$ generated by reflections $s_\alpha : \mathbb{E} \to \mathbb{E}$, $u \mapsto u - \langle u, \alpha^\vee\rangle\alpha$, $\alpha \in \Phi$. If $S := \{s_{\alpha_1}, \cdots, s_{\alpha_n}\}$, then (W, S) is a Coxeter system. Let $\ell : W \to \mathbb{N}$ be the length function on W defined by the set S of simple reflections. If l is a positive integer, the affine Weyl group $W_l = W \ltimes lQ$ is a group of affine transformations of $\mathbb{E}$. If $\alpha \in \Phi, m \in \mathbb{Z}$, let $s_{\alpha,m} : \mathbb{E} \to \mathbb{E}$ be defined by $s_{\alpha,m}(u) = u - (\langle u, \alpha^\vee\rangle - ml)\alpha$. Then $s_{\alpha,m} \in W_l$. If $\alpha_0 \in \Phi$ denotes the maximal short root, (W_l, S_l) is a Coxeter system, setting $S_l := \{s_{\alpha_1}, \cdots, s_{\alpha_n}, s_{\alpha_0,1}\}$. The extended affine Weyl group $\widetilde{W_l}$ is obtained by putting $\widetilde{W_l} = W \ltimes lX$. Although it need not be a Coxeter group, it contains W_l as a normal subgroup satisfying $\widetilde{W_l}/W_l \cong X/Q$. Denote the quotient map $\widetilde{W_l} \to W \cong \widetilde{W_l}/lX$ by $w \mapsto \overline{w}$. The Coxeter number of Φ is $h = \langle \rho, \alpha_0^\vee\rangle + 1$ where $\rho := \varpi_1 + \cdots + \varpi_n = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$ is the Weyl weight. For $w \in W_l$, $x \in \mathbb{E}$, we put $w \cdot x := w(x + \rho) - \rho$. For $J \subseteq \Pi$, let $\Phi_J = \Phi \cap \mathbb{Z}J$ be the root system of Φ generated by J.

Suppose that A and B are augmented algebras (over some common field) and that A is a subalgebra of B. Then A is called normal in B if $BA_+ = A_+B$ where A_+ denotes the augmentation ideal of A. In this situation, put $B//A := B/I$, where $I := BA_+$.

1.3. Main results. Let G be the simple, simply connected algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$ and root system Φ. Let $\mathfrak{t}$ be a fixed maximal toral subalgebra of $\mathfrak{g}$. Given $\alpha \in \Phi$, let $\mathfrak{g}_\alpha$ be the α-root space. Put $\mathfrak{b}^+ = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$ (the positive Borel subalgebra), and $\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi^-} \mathfrak{g}_\alpha$ (the opposite Borel subalgebra).

For $J \subseteq \Pi$, let $\mathfrak{l}_J = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi_J} \mathfrak{g}_\alpha$ be the Levi subalgebra containing $\mathfrak{t}$ and having root system Φ_J. Then $\mathfrak{p}_J = \mathfrak{l}_J \oplus \mathfrak{u}_J \supseteq \mathfrak{b}$ is a parabolic subalgebra of $\mathfrak{g}$, where $\mathfrak{u}_J = \bigoplus_{\alpha\in\Phi^-\backslash\Phi_J^-} \mathfrak{g}_\alpha$

is the nilpotent radical of $\mathfrak{p}_J$. Let $\mathcal{N}(\Phi_J) := G \cdot \mathfrak{u}_J$, where G acts on $\mathfrak{u}_J \subset \mathfrak{g}$ via the adjoint action.

Assumption 1.3.1. *Throughout this paper let $l > 1$ be a fixed odd positive integer. If Φ has type G_2, then we assume that 3 does not divide l. In addition, we assume that l is not a bad prime for Φ. See Section 3.1 for the definition of a bad prime.*

Let
$$\Phi_0 = \Phi_{0,l} := \{\alpha \in \Phi \mid \langle \rho, \alpha^\vee \rangle \equiv 0 \bmod l\}.$$
The restrictions on l guarantee (see Theorem 3.5.1) that there exists $J \subseteq \Pi$ and $w \in W$ such that
$$w(\Phi_0) = \Phi_J.$$
Set $\mathcal{N}(\Phi_0) := \mathcal{N}(\Phi_J)$. If $w' \in W$ and $J' \subseteq \Pi$ also satisfy $w'(\Phi_0) = \Phi_{J'}$, the Johnston-Richardson Theorem [JR] guarantees that $\mathcal{N}(\Phi_0) = \mathcal{N}(\Phi_{J'})$, so that $\mathcal{N}(\Phi_0)$ is well-defined.

Let ζ be a primitive lth root of unity, and let $U_\zeta := U_\zeta(\mathfrak{g})$, $u_\zeta := u_\zeta(\mathfrak{g})$ be the quantum enveloping algebras attached to the complex simple Lie algebra $\mathfrak{g}$ with root system Φ. These algebras were introduced in Section 1.1, and will be developed more fully in Section 2.2 below. The first major result of the paper is the computation of the cohomology algebra $\mathrm{H}^\bullet(u_\zeta, \mathbb{C})$. This result extends the main theorem of Ginzburg-Kumar [GK, Main Thm.] which required that $l > h$.

Theorem 1.3.2. *Let l be as in Assumption 1.3.1, and choose $w \in W$, $J \subseteq \Pi$ such that $w(\Phi_0) = \Phi_J$.*

(a) *We have $\mathrm{H}^{2\bullet+1}(u_\zeta, \mathbb{C}) = 0$, i. e., the cohomology of u_ζ vanishes in all odd degrees.*

(b) *The even degree cohomology algebra $\mathrm{H}^{2\bullet}(u_\zeta, \mathbb{C})$ is described below.*

  (i) *Suppose that $l \nmid n+1$ when Φ is of type A_n and $l \neq 9$ when Φ is of type E_6. Then*
$$\mathrm{H}^{2\bullet}(u_\zeta, \mathbb{C}) \cong \operatorname{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*) \cong \mathbb{C}[G \times_{P_J} \mathfrak{u}_J].$$
  If we assume further that $l \neq 7, 9$ when Φ is of type E_8, then
$$\mathrm{H}^{2\bullet}(u_\zeta, \mathbb{C}) \cong \mathbb{C}[\mathcal{N}(\Phi_0)].$$

  (ii) *If Φ is of type A_n and $l \mid n+1$ with $n+1 = l(m+1)$, then*
$$\mathrm{H}^{2\bullet}(u_\zeta, \mathbb{C}) \cong \operatorname{ind}_{P_J}^G \left(\bigoplus_{t=0}^{l-1} S^{\frac{2\bullet-(m+1)t(l-t)}{2}}(\mathfrak{u}_J^*) \otimes \varpi_{t(m+1)} \right),$$
  where $\varpi_0 = 0$.

  (iii) *If Φ is of type E_6 and $l = 9$, one can take $J = \{\alpha_4\}$. Then*
$$\mathrm{H}^{2\bullet}(u_\zeta, \mathbb{C}) \cong \operatorname{ind}_{P_J}^G \left(S^\bullet(\mathfrak{u}_J^*) \oplus (S^{\frac{2\bullet-12}{2}}(\mathfrak{u}_J^*) \otimes \varpi_1) \oplus (S^{\frac{2\bullet-12}{2}}(\mathfrak{u}_J^*) \otimes \varpi_6) \right).$$

This theorem will be proved in Section 5.5. In order to state our second main result, we need to make a very minor additional restriction on l.

Assumption 1.3.3. *Let $l > 1$ be a fixed odd positive integer satifying Assumption 1.3.1. Moreover assume that when Φ is of type B_n or C_n that $l > 3$.*

We also have the following finite generation result, which is proved in Section 6.

Theorem 1.3.4. *Let l be as in Assumption 1.3.3.*

(a) *The algebra $\mathrm{H}^\bullet(u_\zeta,\mathbb{C})$ is a finitely generated $\mathbb{C}$-algebra.*

(b) *For any finite dimensional $u_\zeta(\mathfrak{g})$-module M, $\mathrm{H}^\bullet(u_\zeta,M)$ is finitely generated as a module for $\mathrm{H}^\bullet(u_\zeta,\mathbb{C})$ (where the action is described in* [PW, Rem. 5.3, Appendix]*).*

Theorem 1.3.4 implies that, given a finite dimensional u_ζ-module M, the support variety $\mathcal{V}_{\mathfrak{g}}(M)$ can be defined as the maximal ideal spectrum $\operatorname{Maxspec}(R/J_M)$, letting J_M be the annihilator in $R := \mathrm{H}^\bullet(u_\zeta(\mathfrak{g}),\mathbb{C})$ for its natural action on $\mathrm{H}^\bullet(u_\zeta(\mathfrak{g}),M\otimes M^*)$ (cf. [PW, §5]). A third major result gives a computation of the support varieties of Weyl modules.

Theorem 1.3.5. *Let l be as in Assumption 1.3.3. Let $\lambda\in X_+$ and let $\mathrm{H}^0_\zeta(\lambda)$ be the induced (or Weyl) module for U_ζ of high weight λ. Suppose that $(l,p)=1$ for any bad prime p of Φ. Then there exists a subset $J\subseteq\Pi$ such that*

$$\mathcal{V}_{\mathfrak{g}}(H^0_\zeta(\lambda)) = G\cdot\mathfrak{u}_J.$$

This result is proved in Section 8.2, where the subset J is explicitly identified in terms of the stabilizer set $\Phi_{\lambda,l}$ (cf. (3.1.1)). In particular, the theorem shows that the support variety of $\mathrm{H}^0_\zeta(\lambda)$ is the closure of a Richardson orbit in $\mathcal{N}(\mathfrak{g})$. In Section 8.3, we show that even when $l>h$, if $p\mid l$ where p is a bad prime for Φ, then $\mathcal{V}_{\mathfrak{g}}(H^0_\zeta(\lambda))$ need not have the form $G\cdot\mathfrak{u}_J$, contrary to the calculations given in [Ost].

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2. Quantum groups, actions, and cohomology

2.1. Listings. For this subsection let Φ be an arbitrary root system with associated Weyl group W. Let $w_0\in W$ be the longest word and $N=|\Phi^+|$. A reduced expression $w_0 = s_{\beta_1}\cdots s_{\beta_N}$ $(\beta_i\in\Pi)$ provides a listing

$$\gamma_1=\beta_1,\gamma_2=s_{\beta_1}(\beta_2),\cdots,\gamma_N=s_{\beta_1}\cdots s_{\beta_{N-1}}(\beta_N)$$

of Φ^+. Define a linear ordering on Φ^+ by setting, for $\beta,\delta\in\Phi^+$,

$$\beta\prec\delta \iff \beta=\gamma_i,\delta=\gamma_j \text{ with } i<j.$$

This ordering depends on the reduced expression chosen for w_0.

Now let $J\subseteq\Pi$, and put $\Phi_J=\Phi\cap\mathbb{Z}J$, the subroot system Φ_J generated by J. Set $W_J=\langle s_\alpha:\alpha\in J\rangle$, the Weyl group of Φ_J, let $w_{0,J}\in W_J$ be the longest word in W_J, and let $w_J := w_{0,J}w_0$. Observe that $w_0=w_{0,J}w_J$ satisfies $\ell(w_0)=\ell(w_{0,J})+\ell(w_J)$. It will often be useful to fix a reduced expression for w_0 by first choosing one for $w_{0,J}$ and then a reduced expression for w_J. If $|\Phi^+_J|=M$, then $\gamma_1\prec\cdots\prec\gamma_M$ lists the positive roots Φ^+_J, while $\gamma_{M+1} \prec \cdots \prec \gamma_N$ lists the remaining positive roots (those in $\Phi^+ \backslash \Phi_J^+$). As above, this ordering depends upon the choice of reduced expressions.

Since $w_{0,J}$ is the longest word in W_J, for $\beta \in J$,

$$\ell(w_{0,J}s_\beta) < \ell(w_{0,J}).$$

Further, w_J is a distinguished right coset representative for W_J in W, so

$$\text{(2.1.1)} \qquad \ell(s_\beta w_J) > \ell(w_J) \qquad \text{and hence} \qquad \ell(s_\beta w_J) = \ell(w_J) + 1.$$

2.2. Quantum enveloping algebras. We return to our assumptions on Φ as in Section 1.2. Let $\mathcal{A} = \mathbb{Q}[q, q^{-1}]$ be the algebra of Laurent polynomials in an indeterminate q. Then $\mathcal{A}$ has fraction field $\mathbb{Q}(q)$. We continue to restrict the integer l according to Assumption 1.3.1. Let $\zeta = \sqrt[l]{1} \in \mathbb{C}$ be a primitive lth root of unity and let $k = \mathbb{Q}(\zeta)$ be the cyclotomic field generated by ζ over $\mathbb{Q}$. We will regard k as an $\mathcal{A}$-algebra by means of the homomorphism $\mathbb{Q}[q, q^{-1}] \to k$, $q \mapsto \zeta$.

The quantized enveloping algebra $\mathbb{U}_q = \mathbb{U}_q(\mathfrak{g})$ of $\mathfrak{g}$ is the $\mathbb{Q}(q)$-algebra with generators $E_\alpha, K_\alpha^{\pm 1}, F_\alpha$, $\alpha \in \Pi$, and relations (R1)—(R6) listed in [Jan2, (4.3)] (whose notation we will generally follow, unless otherwise indicated). The Hopf algebra structure on $\mathbb{U}_q(\mathfrak{g})$ has comultiplication Δ, counit ϵ, and antipode S explicitly described in [Jan2, Ch. 4].

Let $\mathbb{U}_q^+$ denote the subalgebra of $\mathbb{U}_q$ generated by the E_α ($\alpha \in \Pi$), $\mathbb{U}_q^-$ denote the subalgebra of $\mathbb{U}_q$ generated by the F_α ($\alpha \in \Pi$), and $\mathbb{U}_q^0$ denote the subalgebra of $\mathbb{U}_q$ generated by the $K_\alpha^{\pm 1}$ ($\alpha \in \Pi$). Multiplication gives isomorphisms of vector spaces

$$\mathbb{U}_q^+ \otimes \mathbb{U}_q^0 \otimes \mathbb{U}_q^- \xrightarrow{\sim} \mathbb{U}_q \xleftarrow{\sim} \mathbb{U}_q^- \otimes \mathbb{U}_q^0 \otimes \mathbb{U}_q^+.$$

We will use analogous $\{+, -, 0\}$-notation upon specialization to ζ below, and the isomorphisms hold there as well.

The algebra $\mathbb{U}_q$ has two $\mathcal{A}$-forms, $U_q^{\mathcal{A}}$ (due to Lusztig) and $\mathcal{U}_q^{\mathcal{A}}$ (due to De Concini and Kac). After base change to k, they play roles analogous to the hyperalgebra of a reductive group and the universal enveloping algebra of its Lie algebra, respectively.

For a positive integer m and $\alpha \in \Pi$, let $E_\alpha^{(m)} = \frac{E_\alpha^m}{[m]_\alpha^!}$. Here $[m]^! = [m][m-1]\cdots[1]$ with $[i] = \frac{q^i - q^{-i}}{q - q^{-1}}$, while, in general, given $f \in \mathcal{A}$ and $\alpha \in \Pi$, f_α is obtained from f by replacing q by $q_\alpha := q^{d_\alpha}$. Defining $F_\alpha^{(m)}$ similarly, let $U_q^{\mathcal{A}}$ be the $\mathcal{A}$-subalgebra of $\mathbb{U}_q$ generated by the $E_\alpha^{(m)}, F_\alpha^{(m)}$, $K_\alpha^{\pm 1}$ for $m \in \mathbb{N}$, $\alpha \in \Pi$.

Put

$$U_\zeta = U_\zeta(\mathfrak{g}) := k \otimes_{\mathcal{A}} U_q^{\mathcal{A}} / \langle 1 \otimes K_\alpha^l - 1 \otimes 1, \alpha \in \Pi \rangle,$$

where $\langle \cdots \rangle$ means "ideal generated by $\cdots$". The Hopf algebra structure on $\mathbb{U}_q$ induces one on $U_q^{\mathcal{A}}$, and then one on U_ζ. By abuse of notation, let $E_\alpha^{(m)}$, etc., $\alpha \in \Pi$, also denote the elements $1 \otimes E_\alpha^{(m)}$, etc. in U_ζ (note that $E_\alpha^{(1)} = E_\alpha$). Then $E_\alpha^l = F_\alpha^l = 0$ and $K_\alpha^l = 1$ in U_ζ. The elements $E_\alpha, K_\alpha, F_\alpha$, $\alpha \in \Pi$, generate a Hopf subalgebra, denoted $u_\zeta = u_\zeta(\mathfrak{g})$, of U_ζ.

Next, define $\mathcal{U}_q^{\mathcal{A}}$ to be the $\mathcal{A}$-subalgebra of $\mathbb{U}_q$ generated by the $E_\alpha, F_\alpha, K_\alpha^{\pm 1}$. There is an inclusion of $\mathcal{A}$-forms: $\mathcal{U}_q^{\mathcal{A}} \subseteq U_q^{\mathcal{A}}$. Then set $\mathcal{U}_k := k \otimes_{\mathcal{A}} \mathcal{U}_q^{\mathcal{A}}$. Finally, put

$$\mathcal{U}_\zeta = \mathcal{U}_\zeta(\mathfrak{g}) = \mathcal{U}_k / \langle 1 \otimes K_\alpha^l - 1 \otimes 1, \alpha \in \Pi \rangle.$$

The (Hopf) algebra $\mathcal{U}_\zeta$ has a central subalgebra $\mathcal{Z}$ such that $u_\zeta \cong \mathcal{U}_\zeta//\mathcal{Z}$; see [DK] for more details. An explicit description of $\mathcal{Z}$ will be given in Section 2.7.

All the U_ζ-modules M considered in this paper will be assumed to be integrable and type 1. Usually, this will be assumed without mention, but occasionally we repeat it for emphasis. In other words, each E_i, F_j acts locally nilpotently on M. In addition, $M = \bigoplus_{\lambda\in X} M_\lambda$, where M_λ consists of all $v \in M$ such that $\left[{K_\alpha;m \atop n} \right] v = \left[{\langle\lambda,\alpha\rangle+m \atop n} \right]_{q=\zeta^{d_\alpha}} v$. See [Jan2] for an explanation of the $\left[{K_\alpha;m \atop n} \right]$.

2.3. Connections with algebraic groups. The Frobenius morphism is a surjective (Hopf) algebra homomorphism

$$\mathrm{Fr} : U_\zeta \twoheadrightarrow \mathbb{U}(\mathfrak{g}),$$

where $\mathbb{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. If I is the augmentation ideal of u_ζ, then Ker Fr $= IU_\zeta = U_\zeta I$. In other words, u_ζ is normal in U_ζ and $U_\zeta//u_\zeta \cong \mathbb{U}(\mathfrak{g})$. By means of Fr, modules for the Lie algebra $\mathfrak{g}$ can be "pulled back" to give modules for U_ζ: given a $\mathfrak{g}$-module M, $M^{[1]} = \mathrm{Fr}^*M$ denotes the U_ζ module obtained from M by making $x \in U_\zeta$ act as a linear transformation on M by $\mathrm{Fr}(x)$. If M is a locally finite $\mathbb{U}(\mathfrak{g})$-module (e. g., finite dimensional), then $M^{[1]}$ is an integrable, type 1 U_ζ-module. Conversely, a U_ζ-module N on which u_ζ acts trivially can be made into a $\mathbb{U}(\mathfrak{g})$-module via the Frobenius map, so that $N = M^{[1]}$ for a locally finite $\mathbb{U}(\mathfrak{g})$-module M.

Let G be the complex simple, simply connected algebraic group having Lie algebra $\mathfrak{g}$. The category of locally finite $\mathbb{U}(\mathfrak{g})$-modules is naturally isomorphic to the category of rational G-modules.

The set X_+ of dominant weights indexes the irreducible modules for U_ζ. Given $\lambda \in X_+$, let $L_\zeta(\lambda)$ be the irreducible U_ζ-module of high weight λ. Also, let $L(\lambda)$ be the irreducible rational G-module of high weight λ. We may identify $L(\lambda)$ with the finite dimensional irreducible $\mathbb{U}(\mathfrak{g})$-module of high weight λ.

The integral group algebra $\mathbb{Z}X$ has basis denoted $e(\mu)$, $\mu \in X$. Given a finite dimensional U_ζ-module M,

$$\mathrm{ch}\, M = \sum_{\mu\in X} \dim\, M_\mu\ e(\mu) \in \mathbb{Z}X$$

denotes its (formal) character, where M_μ is the dimension of the μ-weight space for the action of U_ζ^0 on M. Sometimes for emphasis we write $\mathrm{ch}_\zeta M$ for $\mathrm{ch}\, M$.

Similarly, if M is a finite dimensional rational G-module (resp. $\mathbb{U}(\mathfrak{g})$-module), let $\mathrm{ch}\, M \in \mathbb{Z}X$ be its formal character with respect to the fixed maximal torus T (resp. Cartan subalgebra $\mathfrak{h} = \mathrm{Lie}(T)$).

2.4. Root vectors and PBW-basis. For each $\alpha \in \Pi$, Lusztig has defined an automorphism T_α of $\mathbb{U}_q$; here we will follow [Jan2, Ch. 8], where the reader can find more details. If $s = s_\alpha \in W$ is the simple reflection defined by α, we write $T_s := T_\alpha$. Given any $w \in W$, let $w = s_{\beta_1}s_{\beta_2}\cdots s_{\beta_n}$ be a reduced expression. Then define $T_w := T_{\beta_1}\cdots T_{\beta_n} \in \mathrm{Aut}(\mathbb{U}_q)$. The automorphism T_w is independent of the reduced expression of w. In other words, the automorphisms T_α extend to an action of the braid group of W on $\mathbb{U}_q$.

Now let $J \subseteq \Pi$ and fix a reduced expression $w_0 = s_{\beta_1} \cdots s_{\beta_N}$ that begins with a reduced expression for $w_{0,J}$ as in Section 2.1. If $w_{0,J} = s_{\beta_1} \cdots s_{\beta_M}$, $s_{\beta_{M+1}} \cdots s_{\beta_N}$ is a reduced expression for $w_J = w_{0,J}w_0$. There is an ordering $\gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_N$ of the positive roots, where $\gamma_i = s_{\beta_1} \cdots s_{\beta_{i-1}}(\beta_i)$. For $\gamma = \gamma_i \in \Phi^+$, the "root vector" $E_\gamma \in \mathbb{U}_q^+$ is defined by

$$E_\gamma = E_{\gamma_i} := T_{s_{\beta_1} \cdots s_{\beta_{i-1}}}(E_{\beta_i}) = T_{\beta_1} \cdots T_{\beta_{i-1}}(E_{\beta_i}).$$

If γ is simple, the "new" E_γ agrees with the original generator E_γ. More generally, E_γ has weight γ. Similarly,

$$F_\gamma = F_{\gamma_i} := T_{s_{\beta_1} \cdots s_{\beta_{i-1}}}(F_{\beta_i}) = T_{\beta_1} \cdots T_{\beta_{i-1}}(F_{\beta_i}),$$

a root vector of weight $-\gamma$.

The subalgebra $\mathbb{U}_q^+$ (resp. $\mathbb{U}_q^-$) has a PBW-like basis consisting of all monomials $E_{\gamma_1}^{a_1} \cdots E_{\gamma_N}^{a_N}$ (resp. $F_{\gamma_1}^{a_1} \cdots F_{\gamma_N}^{a_N}$), $a_1, \cdots, a_n \in \mathbb{N}$. Using divided powers when necessary, one obtains PBW-bases for the specialized quantum groups U_ζ^+, U_ζ^-, u_ζ^+, u_ζ^-, $\mathcal{U}_\zeta^+$, and $\mathcal{U}_\zeta^-$. The automorphisms T_w induce automorphisms on $U_q^{\mathcal{A}}$ and hence on U_ζ.

The monomial bases satisfy a key "commutativity" property originally observed by Levendorskiĭ and Soibelman. Here $E_{\gamma_i}^0 = 1$ and $F_{\gamma_i}^0 = 1$.

Lemma 2.4.1. *([DP, Thm. 9.3], [LS]) In $\mathbb{U}_q$, we have for $i < j$*

(a) $E_{\gamma_i}E_{\gamma_j} = q^{\langle\gamma_i,\gamma_j\rangle}E_{\gamma_j}E_{\gamma_i} + (*)$ *where* $(*)$ *is an $\mathcal{A}$-linear combination of monomials* $E_{\gamma_1}^{a_1} \cdots E_{\gamma_N}^{a_N}$, *with $a_s = 0$ unless $i < s < j$;*

(b) $F_{\gamma_i}F_{\gamma_j} = q^{\langle\gamma_i,\gamma_j\rangle}F_{\gamma_j}F_{\gamma_i} + (*)$ *where* $(*)$ *is an $\mathcal{A}$-linear combination of monomials* $F_{\gamma_1}^{a_1} \cdots F_{\gamma_N}^{a_N}$, *with $a_s = 0$ unless $i < s < j$.*

2.5. Levi and parabolic subalgebras. Given a subset $J \subseteq \Pi$, consider the Levi and parabolic Lie subalgebras $\mathfrak{l}_J$ and $\mathfrak{p}_J = \mathfrak{l}_J \oplus \mathfrak{u}_J$ of $\mathfrak{g}$. We denote the respective universal enveloping algebras by $\mathbb{U}(\mathfrak{l}_J)$ and $\mathbb{U}(\mathfrak{p}_J)$. One can naturally define corresponding quantized enveloping algebras $\mathbb{U}_q(\mathfrak{l}_J)$ and $\mathbb{U}_q(\mathfrak{p}_J)$. As subalgebras of $\mathbb{U}_q$, $\mathbb{U}_q(\mathfrak{l}_J)$ is the subalgebra generated by $\{E_\alpha, F_\alpha : \alpha \in J\} \cup \{K_\alpha^{\pm 1} : \alpha \in \Pi\}$, and $\mathbb{U}_q(\mathfrak{p}_J)$ is the subalgebra generated by $\{E_\alpha : \alpha \in J\} \cup \{F_\alpha, K_\alpha^{\pm 1} : \alpha \in \Pi\}$. For example, if $J = \varnothing$, then $\mathfrak{l}_J = \mathfrak{t}$, $\mathfrak{p}_J = \mathfrak{b}$, $\mathbb{U}_q(\mathfrak{l}_J) = \mathbb{U}_q^0$, and $\mathbb{U}_q(\mathfrak{p}_J) = \mathbb{U}_q(\mathfrak{b}) = \mathbb{U}_q^- \cdot \mathbb{U}_q^0$. Specializing gives the subalgebras $U_\zeta(\mathfrak{l}_J), U_\zeta(\mathfrak{p}_J), u_\zeta(\mathfrak{l}_J), u_\zeta(\mathfrak{p}_J)$ of U_ζ, and $\mathcal{U}_\zeta(\mathfrak{l}_J)$ and $\mathcal{U}_\zeta(\mathfrak{p}_J)$ of $\mathcal{U}_\zeta$.

If we denote by $\mathfrak{p}_J^+ = \mathfrak{l}_J \oplus \mathfrak{u}_J^+$ the opposite parabolic subalgebra (containing the positive Borel subalgebra $\mathfrak{b}^+$), then one can analogously consider $\mathbb{U}_q(\mathfrak{p}_J^+)$.

2.6. The subalgebra $\mathbb{U}_q(\mathfrak{u}_J)$. In this section, we define a subalgebra $\mathbb{U}_q(\mathfrak{u}_J)$ which plays a role analogous to that of $\mathbb{U}(\mathfrak{u}_J) \subset \mathbb{U}(\mathfrak{p}_J)$. As above, choose a reduced expression for w_0 (beginning with one for $w_{0,J}$). Define $\mathbb{U}_q(\mathfrak{u}_J)$ to be the subspace of $\mathbb{U}_q^-$ spanned by the $F_{\gamma_{M+1}}^{a_{M+1}} \cdots F_{\gamma_N}^{a_N}$, $a_i \in \mathbb{N}$. By Lemma 2.4.1, $\mathbb{U}_q(\mathfrak{u}_J)$ is a subalgebra of $\mathbb{U}_q^-$. One can also verify directly that $\mathbb{U}_q(\mathfrak{u}_J)$ is independent of the choice of reduced expression for w_0. However, this follows from a more general set-up which will be useful for other purposes as well.

Given any $v \in W$, we define as in [Jan2, 8.24] a subspace $U^-[v] \subset \mathbb{U}_q^-$ (resp. $U^+[v] \subset \mathbb{U}_q^+$) as follows. Choose a reduced expression $v = s_{\eta_1}s_{\eta_2} \cdots s_{\eta_t}$. For $1 \leq i \leq t$, set $f_i = T_{s_{\eta_1}s_{\eta_2} \cdots s_{\eta_{i-1}}}(F_{\eta_i})$ (resp. $e_i = T_{s_{\eta_1}s_{\eta_2} \cdots s_{\eta_{i-1}}}(E_{\eta_i})$). The f_i (resp. e_i) are in some sense "root vectors" like those defined earlier. Then $U^-[v]$ (resp. $U^+[v]$) is defined to be the span of all monomials of the form $f_1^{a_1} f_2^{a_2} \cdots f_t^{a_t}$ (resp. $e_1^{a_1} e_2^{a_2} \cdots e_t^{a_t}$) for $a_i \geq 0$. By [DKP, 2.2], $U^-[v]$ (resp. $U^+[v]$) is an algebra and moreover is independent of the choice of reduced expression for v.

Since $\mathbb{U}_q(\mathfrak{u}_J) = T_{w_{0,J}}(U^-[w_J])$, $\mathbb{U}_q(\mathfrak{u}_J)$ is a subalgebra of $\mathbb{U}_q(\mathfrak{p}_J) \subset \mathbb{U}_q$, independent of the reduced expression for w_J. Further, since the automorphism $T_{w_{0,J}}$ is also independent of the choice of reduced expression for $w_{0,J}$, the algebra $\mathbb{U}_q(\mathfrak{u}_J)$ depends only on J, not on our above choices of reduced expressions. Following the procedure in Section 2.2, the algebra $\mathbb{U}_q(\mathfrak{u}_J)$ can be specialized to give algebras $U_\zeta(\mathfrak{u}_J)$, $u_\zeta(\mathfrak{u}_J)$, and $\mathcal{U}_\zeta(\mathfrak{u}_J)$. For example, $U_\zeta(\mathfrak{u}_J)$ is the subalgebra of U_ζ^- spanned by $F_{\gamma_{M+1}}^{(a_{M+1})} \cdots F_{\gamma_N}^{(a_N)}$, $a_i \in \mathbb{N}$.

Similarly one can define a subalgebra $\mathbb{U}_q(\mathfrak{u}_J^+) \subset \mathbb{U}_q(\mathfrak{p}_J)$, and the corresponding specializations. Then $\mathbb{U}_q(\mathfrak{u}_J^+) = T_{w_{0,J}}(U^+[w_J])$. When $J = \varnothing$, $\mathbb{U}_q(\mathfrak{u}_J) = \mathbb{U}_q^-$ and $\mathbb{U}_q(\mathfrak{u}_J^+) = \mathbb{U}_q^+$.

2.7. Adjoint Action. Given a Hopf algebra A, the adjoint action of A is defined by setting, for $x, y \in A$, $\mathrm{Ad}(x)(y) = \sum x_1 y S(x_2)$ where $\Delta(x) = \sum x_1 \otimes x_2$ is the comultiplication and S is the antipode. We consider, in particular, the case $A = \mathbb{U}_q$, where we record the formulas here on generators: (for $m \in \mathbb{U}_q$)

$$\begin{cases} \mathrm{Ad}(E_\alpha)(m) & = E_\alpha m - K_\alpha m K_\alpha^{-1} E_\alpha \\ \mathrm{Ad}(K_\alpha^{\pm 1})(m) & = K_\alpha^{\pm 1} m K_\alpha^{\mp 1} \\ \mathrm{Ad}(F_\alpha)(m) & = (F_\alpha m - m F_\alpha) K_\alpha. \end{cases} \tag{2.7.1}$$

By twisting the Hopf structure by the algebra involution ω of $\mathbb{U}_q$ given by $\omega(E_\alpha) = F_\alpha$, $\omega(F_\alpha) = E_\alpha$, and $\omega(K_\alpha) = K_\alpha^{-1}$, we obtain another Hopf structure on $\mathbb{U}_q$ (cf. [Jan2, 3.8]). This procedure then gives an alternate adjoint action $\overline{\mathrm{Ad}}$, which satisfies:

$$\begin{cases} \overline{\mathrm{Ad}}(E_\alpha)(m) & = (E_\alpha m - m E_\alpha) K_\alpha^{-1} \\ \overline{\mathrm{Ad}}(K_\alpha^{\pm})(m) & = K_\alpha^{\pm 1} m K_\alpha^{\mp 1} \\ \overline{\mathrm{Ad}}(F_\alpha)(m) & = F_\alpha m - K_\alpha^{-1} m K_\alpha F_\alpha. \end{cases} \tag{2.7.2}$$

Proposition 2.7.1. *The following stability results hold:*

(a) *The subalgebra* $\mathbb{U}_q(\mathfrak{u}_J^+)$ *is stable under the* Ad *action of* $\mathbb{U}_q(\mathfrak{p}_J^+)$ *on itself.*
(b) *The subalgebra* $\mathbb{U}_q(\mathfrak{u}_J)$ *is stable under the* $\overline{\mathrm{Ad}}$ *action of* $\mathbb{U}_q(\mathfrak{p}_J)$ *on itself.*

Proof. We prove part (a). Part (b) follows from part (a) since, for $x \in \mathbb{U}_q(\mathfrak{p}_J)$, $\overline{\mathrm{Ad}}(x) = \omega \circ \mathrm{Ad}(\omega(x)) \circ \omega$. Since $\mathbb{U}_q(\mathfrak{p}_J^+)$ is generated as an algebra by $\{E_\alpha, K_\alpha, K_\alpha^{-1} : \alpha \in \Pi\} \cup \{F_\alpha : \alpha \in J\}$, it suffices to show that $\mathbb{U}_q(\mathfrak{u}_J^+)$ is stable under the action of these elements. By (2.7.1), $\mathrm{Ad}(K_\alpha)$ and $\mathrm{Ad}(K_\alpha^{-1})$ simply act by scalar multiplication on the weight spaces which span $\mathbb{U}_q(\mathfrak{u}_J^+)$. Hence $\mathbb{U}_q(\mathfrak{u}_J^+)$ is stable under $\mathrm{Ad}(K_\alpha^{\pm 1})$. For $\alpha \in \Phi^+ \backslash \Phi_J^+$, the stability under $\mathrm{Ad}(E_\alpha)$ follows from the fact that $\mathbb{U}_q(\mathfrak{u}_J^+)$ is an algebra. It remains to prove stability under $\mathrm{Ad}(F_\alpha)$ and $\mathrm{Ad}(E_\alpha)$ for $\alpha \in J$. Fix $\alpha \in J$ and set $\beta = -w_{0,J}(\alpha) \in J$.

By Section 2.6, $\mathbb{U}_q(\mathfrak{u}_J^+) = T_{w_{0,J}}(U^+[w_J])$. Given a reduced expression $w_J = s_{\eta_1} s_{\eta_2} \cdots s_{\eta_t}$, $U^+[w_J]$ is spanned by monomials of ordered root vectors:

$$E_{\eta_1}, T_{s_{\eta_1}}(E_{\eta_2}), \dots, T_{s_{\eta_1} \cdots s_{\eta_{t-1}}}(E_{\eta_t}).$$

Since $w_J = w_{0,J} w_0$, $w_J^{-1}(\beta) \in \Pi$. Consider also $U^+[w_J s_{w_J^{-1}(\beta)}]$. Since $w_J s_{w_J^{-1}(\beta)} = s_\beta w_J$, by (2.1.1), $\ell(w_J s_{w_J^{-1}(\beta)}) = \ell(w_J) + 1$. Therefore, the ordered root vectors defining $U^+[w_J s_{w_J^{-1}(\beta)}]$

are

$$E_{\eta_1}, T_{s_{\eta_1}}(E_{\eta_2}), \dots, T_{s_{\eta_1}\cdots s_{\eta_{t-1}}}(E_{\eta_t}), T_{w_J}(E_{w_J^{-1}(\beta)}).$$

By [Jan2, Prop. 8.20] $T_{w_J}(E_{w_J^{-1}(\beta)}) = E_{w_J(w_J^{-1}(\beta))} = E_\beta$. Now $U^+[w_J]$ is a subalgebra of $U^+[w_J s_{w_J^{-1}(\beta)}]$ spanned by monomials not involving the last root vector E_β. Moreover, by Lemma 2.4.1, since E_β appears last in the ordering of root vectors in $U^+[w_J s_{w_J^{-1}(\beta)}]$,

$$\text{if } u \in U^+[w_J] \subset U^+[w_J s_{w_J^{-1}(\beta)}] \text{ is a monomial, then } uE_\beta - q^{\langle \mathrm{wt}(u), \beta\rangle} E_\beta u \in U^+[w_J], \tag{2.7.3}$$

where $\mathrm{wt}(u)$ denotes the weight of u.

Now consider the subalgebra $U^+[s_\beta w_J]$. The ordered root vectors defining $U^+[s_\beta w_J]$ are

$$E_\beta, T_{s_\beta}(E_{\eta_1}), T_{s_\beta}(T_{s_{\eta_1}}(E_{\eta_2})), \dots, T_{s_\beta}(T_{s_{\eta_1}\cdots s_{\eta_{t-1}}}(E_{\eta_t})).$$

Then $T_{s_\beta}(U^+[w_J])$ is evidently a subalgebra of $U^+[s_\beta w_J]$ spanned by monomials in all but the first root vector E_β. Moreover, by Lemma 2.4.1, since E_β occurs first in the root ordering,

$$\text{if } u \in U^+[w_J] \text{ is a monomial, then } E_\beta T_{s_\beta}(u) - q^{\langle \mathrm{wt}(T_{s_\beta}(u)), \beta\rangle} T_{s_\beta}(u) E_\beta \in T_{s_\beta}(U^+[w_J]). \tag{2.7.4}$$

We now show that $\mathrm{Ad}(F_\alpha)$ preserves $\mathbb{U}_q(\mathfrak{u}_J^+)$. Since $\alpha, \beta \in \Pi$, by [Jan2, Prop. 8.20], $F_\alpha = T_{w_{0,J}s_\beta}(F_\beta)$. By [Jan2, 8.14(3)], $F_\beta = -T_{s_\beta}(E_\beta)K_\beta^{-1} = -T_{s_\beta}(E_\beta K_\beta)$. Combining these gives $F_\alpha = T_{w_{0,J}s_\beta}(-T_{s_\beta}(E_\beta K_\beta)) = -T_{w_{0,J}}(E_\beta K_\beta)$.

Let $T_{w_{0,J}}(u)$ for $u \in U^+[w_J]$ be an arbitrary monomial element of $\mathbb{U}_q(\mathfrak{u}_J^+)$. Then we have

$$\begin{aligned}
\mathrm{Ad}(F_\alpha)(T_{w_{0,J}}(u)) &= (F_\alpha T_{w_{0,J}}(u) - T_{w_{0,J}}(u) F_\alpha) K_\alpha && \text{by (2.7.1)}\\
&= (-T_{w_{0,J}}(E_\beta K_\beta) T_{w_{0,J}}(u) + T_{w_{0,J}}(u) T_{w_{0,J}}(E_\beta K_\beta)) K_\alpha && \text{from above}\\
&= T_{w_{0,J}}((uE_\beta K_\beta - E_\beta K_\beta u) K_\beta^{-1}) && \text{[Jan2, 8.18(3)]}\\
&= T_{w_{0,J}}(uE_\beta - E_\beta K_\beta u K_\beta^{-1}) &&\\
&= T_{w_{0,J}}(uE_\beta - q^{\langle \mathrm{wt}(u), \beta\rangle} E_\beta u) &&\\
&\in T_{w_{0,J}}(U^+[w_J]) = \mathbb{U}_q(\mathfrak{u}_J^+) && \text{by (2.7.3)}
\end{aligned}$$

as claimed.

Lastly, we consider E_α. Again $\alpha, \beta \in \Pi$, so $E_\alpha = T_{w_{0,J}s_\beta}(E_\beta)$. Let $T_{w_{0,J}}(u)$ for $u \in U^+[w_J]$ be an arbitrary monomial element of $\mathbb{U}_q(\mathfrak{u}_J^+)$. Then we have

$$\begin{aligned}
\mathrm{Ad}(E_\alpha)(T_{w_{0,J}}(u)) &= E_\alpha T_{w_{0,J}}(u) - K_\alpha T_{w_{0,J}}(u) K_\alpha^{-1} E_\alpha && \text{by (2.7.1)}\\
&= T_{w_{0,J}s_\beta}(E_\beta) T_{w_{0,J}}(u) - K_\alpha T_{w_{0,J}}(u) K_\alpha^{-1} T_{w_{0,J}s_\beta}(E_\beta) && \text{from above}\\
&= T_{w_{0,J}s_\beta}(E_\beta T_{s_\beta}(u) - K_\beta T_{s_\beta}(u) K_\beta^{-1} E_\beta) && \text{[Jan2, 8.18(3)]}\\
&= T_{w_{0,J}s_\beta}(E_\beta T_{s_\beta}(u) - q^{\langle \mathrm{wt}(T_{s_\beta}(u)), \beta\rangle} T_{s_\beta}(u) E_\beta) &&\\
&\in T_{w_{0,J}s_\beta}(T_{s_\beta}(U^+[w_J])) = T_{w_{0,J}}(U^+[w_J]) = \mathbb{U}_q(\mathfrak{u}_J^+) && \text{by (2.7.4)}
\end{aligned}$$

as claimed. □

The definitions of Ad and $\overline{\mathrm{Ad}}$ now give the following.

Corollary 2.7.2. *The algebra $\mathbb{U}_q(\mathfrak{u}_J)$ is normal in $\mathbb{U}_q(\mathfrak{p}_J)$ and the algebra $\mathbb{U}_q(\mathfrak{u}_J^+)$ is normal in $\mathbb{U}_q(\mathfrak{p}_J^+)$. Further, normality also holds for the specializations $U_\zeta(\mathfrak{u}_J) \subset U_\zeta(\mathfrak{p}_J)$, $u_\zeta(\mathfrak{u}_J) \subset u_\zeta(\mathfrak{p}_J)$, and $\mathcal{U}_\zeta(\mathfrak{u}_J) \subset \mathcal{U}_\zeta(\mathfrak{p}_J)$ (as well as for the $+$-versions). Also, $U_\zeta(\mathfrak{l}_J) \cong U_\zeta(\mathfrak{p}_J)//U_\zeta(\mathfrak{u}_J)$, $u_\zeta(\mathfrak{l}_J) \cong u_\zeta(\mathfrak{p}_J)//u_\zeta(\mathfrak{u}_J)$, and $\mathcal{U}_\zeta(\mathfrak{l}_J) \cong \mathcal{U}_\zeta(\mathfrak{p}_J)//\mathcal{U}_\zeta(\mathfrak{u}_J)$.*

Recall that we have an inclusion of $\mathcal{A}$-forms $\mathcal{U}_q^{\mathcal{A}} \subseteq U_q^{\mathcal{A}}$. While it is generally false that $\mathcal{U}_q^{\mathcal{A}}$ is stable under the adjoint action of $U_q^{\mathcal{A}}$ on itself, this property does hold after base-change from $\mathcal{A}$ to a larger algebra $\mathcal{B}$. More precisely, given l, let $f_l(x) \in \mathbb{Q}[x]$ denote the minimal polynomial for a primitive lth root of unity. Set $\mathcal{B} := \mathbb{Z}[q, q^{-1}]_{(\langle f_l(q) \rangle)}$, i. e., the local ring determined by the maximal ideal $\langle f_l(q) \rangle$. Define $U_q^{\mathcal{B}}$ and $\mathcal{U}_q^{\mathcal{B}}$ analogous to the way $U_q^{\mathcal{A}}$ and $\mathcal{U}_q^{\mathcal{A}}$ were defined in Section 2.2. Then we still have an inclusion $\mathcal{U}_q^{\mathcal{B}} \subset U_q^{\mathcal{B}}$.

Lemma 2.7.3. [ABG, Prop. 2.9.2(i)] *The adjoint action of $U_q^{\mathcal{B}}$ stabilizes $\mathcal{U}_q^{\mathcal{B}}$. That is,*

$$\operatorname{Ad}(U_q^{\mathcal{B}})(\mathcal{U}_q^{\mathcal{B}}) \subseteq \mathcal{U}_q^{\mathcal{B}} \qquad \text{and} \qquad \overline{\operatorname{Ad}}(U_q^{\mathcal{B}})(\mathcal{U}_q^{\mathcal{B}}) \subseteq \mathcal{U}_q^{\mathcal{B}}.$$

Hence, the adjoint action (either Ad *or* $\overline{\operatorname{Ad}}$*) of U_ζ defines an action on $\mathcal{U}_\zeta$.*

Let $\mathcal{Z} \subset \mathcal{U}_\zeta$ be the central subalgebra such that $\mathcal{U}_\zeta//\mathcal{Z} \cong u_\zeta$ (cf. Section 2.2). In terms of root vectors, $\mathcal{Z}$ is the algebra generated by $\{E_\gamma^l, F_\gamma^l : \gamma \in \Phi^+\}$. Given J, define a central subalgebra $Z_J \subset \mathcal{U}_\zeta(\mathfrak{u}_J)$ as $Z_J := \mathcal{Z} \cap \mathcal{U}_\zeta(\mathfrak{u}_J)$. Clearly Z_J is the subalgebra generated by $\{F_\gamma^l : \gamma \in \Phi^+ \backslash \Phi_J^+\}$. Further, we see that $\mathcal{U}_\zeta(\mathfrak{u}_J)//Z_J \cong u_\zeta(\mathfrak{u}_J)$. We now obtain the following generalization of [ABG, Prop. 2.9.2].

Corollary 2.7.4. *Under the induced $\overline{\operatorname{Ad}}$-action of U_ζ on $\mathcal{U}_\zeta$ we have the following:*

(a) $\overline{\operatorname{Ad}}(U_\zeta(\mathfrak{p}_J))$ *stabilizes $\mathcal{U}_\zeta(\mathfrak{u}_J)$, Z_J, and $u_\zeta(\mathfrak{u}_J)$.*
(b) *The action of $u_\zeta(\mathfrak{p}_J)$ on Z_J is trivial.*
(c) $\overline{\operatorname{Ad}}$ *induces an action of $\mathbb{U}(\mathfrak{p}_J)$ on Z_J.*

2.8. Finite dimensionality of cohomology groups.

For any field k, given augmented k-algebras $A \subset B$ with A normal in B (cf. Sections 1.2 and 2.7), there exists a Lyndon-Hochschild-Serre (LHS) spectral sequence (cf. [GK, §5.2, 5.3]). More precisely, we have the following result, which will be used below.

Lemma 2.8.1. *Assume that B is flat as a right A-module. For any left B-module M, there is a natural action of $B//A$ on $\mathrm{H}^\bullet(A, M)$ which gives rise to a spectral sequence*

$$E_2^{i,j} = \mathrm{H}^i(B//A, \mathrm{H}^j(A, M)) \Rightarrow \mathrm{H}^{i+j}(B, M).$$

We return to the situation of the previous section. Under the $\overline{\operatorname{Ad}}$-action of $U_\zeta(\mathfrak{p}_J)$ on itself, $U_\zeta(\mathfrak{p}_J)$ admits the structure of a Hopf module algebra (cf. [Mon, 4.1.1, 4.1.9]). The action of $U_\zeta(\mathfrak{p}_J)$ on $\mathcal{U}_\zeta(\mathfrak{p}_J)$ can be extended to an action on the bar resolution which computes the cohomology $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{p}_J), \mathbb{C})$. Thus, there is a natural action of $U_\zeta(\mathfrak{p}_J)$ on $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{p}_J), \mathbb{C})$ and further on $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})$. In particular, U_ζ^0 acts on these cohomology spaces.

Lemma 2.8.2. *For each nonnegative integer n, $\mathrm{H}^n(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})$ is a finite dimensional.*

Proof. If Z_J is the central subalgebra of $\mathcal{U}_\zeta(\mathfrak{u}_J)$ defined after Lemma 2.7.3, $\mathcal{U}_\zeta(\mathfrak{u}_J)//Z_J \cong u_\zeta(\mathfrak{u}_J)$. By Lemma 2.8.1, there is a spectral sequence

$$E_2^{i,j} = \mathrm{H}^i(u_\zeta(\mathfrak{u}_J), \mathrm{H}^j(Z_J, \mathbb{C})) \Rightarrow \mathrm{H}^{i+j}(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C}).$$

The subalgebra Z_J is central so it follows from [GK, Lemma 5.2.2] that the action of $u_\zeta(\mathfrak{u}_J)$ on $\mathrm{H}^\bullet(Z_J, \mathbb{C})$ is trivial, and

$$E_2^{\bullet,\bullet} \cong \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}_J), \mathbb{C}) \otimes \mathrm{H}^\bullet(Z_J, \mathbb{C}).$$

Since Z_J is the symmetric algebra based on the vector space $\mathfrak{u}_J^{[1]}$, we get that $\mathrm{H}^\bullet(Z_J, \mathbb{C}) \cong \Lambda^\bullet(\mathfrak{u}_J^*)^{[1]}$. Moreover, $u_\zeta(\mathfrak{u}_J)$ is finite dimensional, so $\mathrm{H}^i(u_\zeta(\mathfrak{u}_J), \mathbb{C})$ is finite dimensional for any integer i. The result follows because the cohomology $\mathrm{H}^n(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})$ is a subquotient of $\oplus_{i+j=n} E_2^{i,j}$ which is finite dimensional. □

Also, the cohomology of $\mathcal{U}_\zeta(\mathfrak{u}_J)$ can be computed by means of the reduced bar resolution (cf. [Mac, Ch. X, §2]). Although it is not clear from this point of view that the cohomology is finite dimensional in any homological degree, it does have a natural U_ζ^0-action induced from the $\overline{\mathrm{Ad}}$-action on $U_\zeta(\mathfrak{u}_J)$. But then the above lemma establishes it is finite dimensional, and this fact implies each cohomology space is a weight module for U_ζ^0. We summarize this result as follows.

Corollary 2.8.3. *The cohomology* $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})$ *is a weight module for* U_ζ^0. *For any* $\lambda \in X$, $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})_\lambda$ *is finite dimensional.*

Proof. It remains only to establish the last statement. But if $\bar{B}_\bullet$ is the reduced bar-resolution, then, given any weight μ, for large n, $(\bar{B}_n)_\mu = 0$. □

2.9. Spectral sequences and Euler characteristic. We study the cohomology of $\mathcal{U}_\zeta(\mathfrak{u}_J)$ and $u_\zeta(\mathfrak{u}_J)$. To do this, we use a filtration on $\mathcal{U}_\zeta(\mathfrak{u}_J)$ (resp. $u_\zeta(\mathfrak{u}_J)$) introduced for $\mathcal{U}_\zeta$ in [DK].

Let $A = \mathcal{U}_\zeta(\mathfrak{u}_J)$. By Section 2.6, A has a basis of monomial elements

$$\{F_{\gamma_{M+1}}^{a_{M+1}} F_{\gamma_{M+2}}^{a_{M+2}} \cdots F_{\gamma_N}^{a_N} : a_i \in \mathbb{N}\},$$

where $N = |\Phi^+|$ and $M = |\Phi_J^+|$. For $\overline{a} := (a_{M+1}, \cdots, a_N) \in \mathbb{N}^{N-M}$, set

$$F_{\overline{a}} := F_{\gamma_{M+1}}^{a_{M+1}} F_{\gamma_{M+2}}^{a_{M+2}} \cdots F_{\gamma_N}^{a_N}.$$

Place a total (lexicographical) ordering $\prec$ on $\mathbb{N}^{N-M}$ as follows. Put $\overline{a} \prec \overline{b}$ if and only if there exists $M+1 \leq i \leq N$ such that $a_i < b_i$ and $a_j = b_j$ for all $j > i$. With this ordering, one can define an $\mathbb{N}^{N-M}$-filtration on A. Given $\overline{a} \in \mathbb{N}^{N-M}$, let $A_{\overline{a}}$ be the subalgebra of A generated by

$$\{F_{\overline{b}} : \overline{b} \preceq \overline{a}\}.$$

By Lemma 2.4.1, this is a multiplicative filtration on A. That is, $A_{\overline{a}} \cdot A_{\overline{b}} \subseteq A_{\overline{a}+\overline{b}}$.

Further, by Lemma 2.4.1 again, the associated graded algebra $\operatorname{gr} A$ is generated by $\{X_\alpha : \alpha \in \Phi^+ \backslash \Phi_J^+\}$ subject to the relations:

$$X_\alpha \cdot X_\beta = \zeta^{\langle \alpha, \beta \rangle} X_\beta \cdot X_\alpha \quad \text{if } \alpha \prec \beta.$$

This filtration induces a filtration on $u_\zeta(\mathfrak{u}_J)$ such that the algebra $\operatorname{gr} u_\zeta(\mathfrak{u}_J)$ is also generated by $\{X_\alpha : \alpha \in \Phi^+ \backslash \Phi_J^+\}$ subject to the above relations, in addition to the condition:

$$X_\alpha^l = 0 \quad \text{for } \alpha \in \Phi^+ \backslash \Phi_J^+.$$

Let $\Lambda_{\zeta,J}^\bullet$ be the graded algebra with generators $\{x_\alpha : \alpha \in \Phi^+ \backslash \Phi_J^+\}$ where $\deg(x_\alpha) = 1$ subject to the following relations:

$$x_\alpha \cdot x_\beta + \zeta^{-\langle \alpha, \beta \rangle} x_\beta \cdot x_\alpha = 0 \quad \text{if } \alpha \prec \beta;$$

$$x_\alpha^2 = 0 \quad \text{for } \alpha \in \Phi^+ \backslash \Phi_J^+.$$

There exists a graded (by degree) algebra isomorphism $\mathrm{H}^\bullet(\mathrm{gr}\,\mathcal{U}_\zeta(\mathfrak{u}_J),\mathbb{C}) \cong \Lambda^\bullet_{\zeta,J}$ (cf. [GK, Prop. 2.1]). This is also an isomorphism of U^0_ζ-modules, where $\Lambda^\bullet_{\zeta,J}$ is regarded as a U^0_ζ-module by assigning x_α weight α.

Proposition 2.9.1. *(a) In the character group* $\mathbb{Z}X$, *we have*

$$\sum_{n=0}^{\infty}(-1)^n \operatorname{ch}\mathrm{H}^n(\mathcal{U}_\zeta(\mathfrak{u}_J),\mathbb{C}) = \sum_{n=0}^{\infty}(-1)^n \operatorname{ch}\Lambda^n_{\zeta,J}.$$

(b) If $\lambda \in X$ *is a weight of* U^0_ζ *in* $\mathrm{H}^n(\mathcal{U}_\zeta(\mathfrak{u}_J),\mathbb{C})$, *then* λ *is a weight of* U^0_ζ *in* $\Lambda^n_{\zeta,J}$.

Proof. Let $A = \mathcal{U}_\zeta(\mathfrak{u}_J)$. By Corollary 2.8.3 and [GK, Prop. 2.1] both $\mathrm{H}^\bullet(A,\mathbb{C})$ and $\mathrm{H}^\bullet(\mathrm{gr}\ A,\mathbb{C})$ have weight space decompositions with finite dimensional weight spaces. Let A_+ and gr A_+ denote the augmentation ideals of A and gr A, respectively. Let $C^\bullet(A)$ and $C^\bullet(\mathrm{gr}\ A)$ be the complexes obtained by taking duals of the respective reduced bar resolutions. More precisely, $C^n(A) = \mathrm{Hom}_{\mathbb{C}}((A_+)^{\otimes n},\mathbb{C})$ and $C^n(\mathrm{gr}\ A) = \mathrm{Hom}_{\mathbb{C}}((\mathrm{gr}\ A_+)^{\otimes n},\mathbb{C})$. Note that A_+ and gr A_+ are isomorphic as U^0_ζ-modules. The same holds for $C^n(A)$ and $C^n(\mathrm{gr}\ A)$ and the differentials of both complexes are U^0_ζ-module maps. Thus, for a weight λ, $\mathrm{H}^\bullet(A,\mathbb{C})_\lambda$ and $\mathrm{H}^\bullet(\mathrm{gr}\ A,\mathbb{C})_\lambda$ identify with the cohomologies of the complexes $C^\bullet(A)_\lambda$ and $C^\bullet(\mathrm{gr}\ A)_\lambda$, respectively.

By the Euler-Poincaré principle (cf. [GW, Lemma 7.3.11]),

$$\begin{aligned}\chi(\mathrm{H}^\bullet(A,\mathbb{C})_\lambda) &:= \sum_{n\geq 0}^{\infty}(-1)^n \dim \mathrm{H}^n(A,\mathbb{C})_\lambda = \sum_{n=0}^{\infty}(-1)^n \dim C^n(A)_\lambda \\ &= \sum_{n=0}^{\infty}(-1)^n \dim C^n(\mathrm{gr}\ A)_\lambda = \sum_{n=0}^{\infty}(-1)^n \dim \mathrm{H}^n(\mathrm{gr}\ A,\mathbb{C})_\lambda := \chi(\mathrm{H}^\bullet(\mathrm{gr}\ A)_\lambda).\end{aligned}$$

Part (a) follows now from the cohomology calculation for gr A given in [GK, Prop. 2.1].

Let $A_\bullet$ be the increasing filtration on A indexed by $\Lambda := \mathbb{N}^{N-M}$, viewed as a poset using the lexicographic ordering $\prec$ above. It induces a (downward) filtration on the complex $C^\bullet(A)_\lambda$ as follows. For $\gamma,\eta \in \Lambda$, set $A_{+\gamma} = A_\gamma \cap A_+$, and define

$$B^n_{[\prec\eta]} = \sum_{\sum\gamma_i\prec\eta} A_{+\gamma_1}\otimes A_{+\gamma_2}\otimes\cdots\otimes A_{+\gamma_n}, \qquad B^n_{[\preceq\eta]} = \sum_{\sum\gamma_i\preceq\eta} A_{+\gamma_1}\otimes A_{+\gamma_2}\otimes\cdots\otimes A_{+\gamma_n}.$$

Observe that $B^n_{[\prec\eta]} \subseteq B^n_{[\preceq\eta]}$, so setting, for $\lambda \in X(T)$,

$$C^n(A)_{\lambda,[\prec\eta]} = \mathrm{Hom}_{\mathbb{C}}(A_+^{\otimes n}/B^n_{[\prec\eta]},\mathbb{C})_\lambda, \qquad C^n(A)_{\lambda,[\preceq\eta]} = \mathrm{Hom}_{\mathbb{C}}(A_+^{\otimes n}/B^n_{[\preceq\eta]},\mathbb{C})_\lambda$$

then $C^n(A)_{\lambda,[\preceq\eta]} \subseteq C^n(A)_{\lambda,[\prec\eta]}$. Moreover, if $\eta,\zeta \in \Lambda$ with $\zeta \prec \eta$ then $C^n(A)_{\lambda,[\prec\eta]} \subseteq C^n(A)_{\lambda,[\prec\zeta]}$.

The grading on gr A leads in a natural way to a grading of the complex $C^\bullet(\mathrm{gr}\ A)_\lambda$. For $\eta \in \Lambda$, $C^\bullet(\mathrm{gr}\,A)_{\lambda,[\eta]}$ denotes the graded component corresponding to η, and we can identify $C^\bullet(A)_{\lambda,[\prec\eta]}/C^\bullet(A)_{\lambda,[\preceq\eta]}$ with $C^\bullet(\mathrm{gr}\ A)_{\lambda,[\eta]}$. Also,

$$C^\bullet(\mathrm{gr}\ A)_\lambda \cong \bigoplus_{\eta\in\Lambda} C^\bullet(A)_{\lambda,[\prec\eta]}/C^\bullet(A)_{\lambda,[\preceq\eta]}.$$

For a fixed weight λ, $C^n(\mathrm{gr}\ A)_\lambda \neq 0$, for finitely many n, and $C^\bullet(A)_{\lambda,[\prec\eta]}/C^\bullet(A)_{\lambda,[\preceq\eta]} \neq 0$ for finitely many η. Let $\overline{\Lambda} = \{\eta \in \Lambda : \ C^n_{\lambda,[\prec\eta]}/C^n_{\lambda,[\preceq\eta]} \neq 0, \text{ for some } n\}$. Then $\overline{\Lambda}$ is a finite totally ordered set in $\mathbb{N}^{N-M}$ which induces a filtration (which can be indexed by $\mathbb{N}$) on $C^\bullet(\text{gr } A)_\lambda$. Therefore, we have a spectral sequence,

$$E_1^{i,j} = (\mathrm{H}^{i+j}(\text{gr } A, \mathbb{C})_\lambda)_{(i)} \Rightarrow \mathrm{H}^{i+j}(A, \mathbb{C})_\lambda.$$

This shows that $\mathrm{H}^n(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})_\lambda$ is a subquotient of $(\Lambda^n_{\zeta,J})_\lambda$, and part (b) follows.

□

2.10. Induction functors. Let $\mathcal{C}$ (resp. $\mathcal{C}^{\leq}$) be the category of type 1, integrable representations of U_ζ (resp. $U_\zeta(\mathfrak{b})$). The restriction functor res : $\mathcal{C} \to \mathcal{C}^{\leq}$ has a right adjoint induction functor $\mathrm{H}^0_\zeta = \mathrm{H}^0(U_\zeta/U_\zeta(\mathfrak{b})) : \mathcal{C}^{\leq} \to \mathcal{C}$ defined by

$$\mathrm{H}^0_\zeta(M) = (M \otimes k[U_\zeta])^{U_\zeta(\mathfrak{b})} \cong \mathcal{F}(\mathrm{Hom}_{U_\zeta(\mathfrak{b})}(U_\zeta, M)).$$

In this expression $k[U_\zeta]$ denotes the coordinate algebra of U_ζ. Also, the functor $\mathcal{F}(-)$ assigns to any U_ζ-module the largest type 1, integrable submodule. We refer to [APW, (2.8), (2.10)] and [RH, (2.9)] for further discussion and explanation of notation.

For $\lambda \in X_+$, the induced module

$$\nabla_\zeta(\lambda) := \mathrm{H}^0(U_\zeta/U_\zeta(\mathfrak{b}), \lambda)$$

has irreducible socle isomorphic to $L_\zeta(\lambda)$. In addition, there is an equality

$$\mathrm{ch}\, \nabla_\zeta(\lambda) = \mathrm{ch}\, L(\lambda) = \sum_{x \in W} (-1)^{\ell(x)} e(w \cdot \lambda) / \sum_{x \in W} (-1)^{\ell(x)} e(x \cdot 0)$$

of formal characters, which is given by Weyl's character formula.

We will also use the induction functors $\mathrm{H}^0(U_\zeta/U_\zeta(\mathfrak{p}_J), -)$ (resp. $\mathrm{H}^0(U_\zeta(\mathfrak{p}_J)/U_\zeta(\mathfrak{b}), -)$) from the category of type 1, integrable $U_\zeta(\mathfrak{p}_J)$-modules (resp. $U_\zeta(\mathfrak{b})$-modules) to type 1, integrable U_ζ-modules (resp. $U_\zeta(\mathfrak{p}_J)$-modules). Note that if λ is a one-dimensional $U_\zeta(\mathfrak{b})$-module then $\mathrm{H}^0(U_\zeta(\mathfrak{p}_J)/U_\zeta(\mathfrak{b}), \lambda)$ is trivial as a $U_\zeta(\mathfrak{u}_J)$-module.

Let $(X_J)_+ \subseteq X$ be the set of J-dominant weights, i. e., $\lambda \in X$ belongs to $(X_J)_+$ if and only if $\langle \lambda, \alpha^\vee \rangle \in \mathbb{N}$ for all $\alpha \in J$. The set $(X_J)_+$ indexes the irreducible (type 1, integrable) $U_\zeta(\mathfrak{l}_J)$-modules. For $\lambda \in (X_J)_+$,

$$\nabla_{J,\zeta}(\lambda) := \mathrm{H}^0(U_\zeta(\mathfrak{p}_J)/U_\zeta(\mathfrak{b}), \lambda)$$

has irreducible socle isomorphic to $L_{J,\zeta}(\lambda)$, the irreducible $U_\zeta(\mathfrak{l}_J)$-module of high weight λ.

If $\lambda \in (X_J)_+$ satisfies $\langle \lambda + \rho, \alpha^\vee \rangle = \ell - 1$ for all $\alpha \in J$ we call λ a J-Steinberg weight. Then $\nabla_{J,\zeta}(\lambda)$ is a projective (and injective) irreducible $U_\zeta(\mathfrak{l}_J)$-module (in the category of type 1, integrable modules). It remains irreducible, projective, and injective upon restriction to $\mathfrak{u}_\zeta(\mathfrak{l}_J)$.

Finally, it will often be convenient to write $\mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)}$ in place of $\mathrm{H}^0(U_\zeta(\mathfrak{p}_J)/U_\zeta(\mathfrak{b}), -)$.

3. Computation of Φ_0 and $\mathcal{N}(\Phi_0)$

3.1. Recall that a prime p is called *bad* for the root system Φ provided that there exists a closed subsystem Φ' of Φ such that Q/Q' has p-torsion, where $Q' = \mathbb{Z}\Phi'$ is the root lattice of Φ'. If p is not bad, then p is called a *good* prime for Φ. Equivalently, p is good if and only if p does not appear as the coefficient of a simple root in the decomposition of $\alpha \in \Phi$

as an integral linear combination of simple roots. The good primes for the various types of irreducible root systems are given as follows:

- Φ of type A_n, all p;
- Φ of type B_n, C_n, D_n, $p \geq 3$;
- Φ of type E_6, E_7, F_4, G_2, $p \geq 5$;
- Φ of type E_8, $p \geq 7$.

Now let $l > 1$ be an integer. We will say that l is good for Φ provided that l is not divisible by a bad prime for Φ. Otherwise, l is bad for Φ. Additionally, a good integer l is said to be *very good* for Φ provided that if Φ has type A_n, then l and $n+1$ are relatively prime. In cases when the root system is not irreducible, we will say that l is good (resp. very good) provided that it is good (resp. very good) for every irreducible component of Φ.

In this paper, the significance of good integers l comes about because of certain closed subsystems $\Phi_{\lambda,l}$ constructed from weights $\lambda \in X$. Precisely, define

$$\Phi_{\lambda,l} = \{\alpha \in \Phi \mid \langle \lambda + \rho, \alpha \rangle \equiv 0 \bmod l\}.$$

In practice, when l is clear from context, we will write Φ_λ rather than $\Phi_{\lambda,l}$. Obviously, Φ_λ is a closed subroot system of Φ. By our assumptions on l, each d_α is relatively prime to l (i. e., l is odd in types B_n, C_n and F_4, and, in type G_2, l is not divisible by 3) so we have

$$\Phi_\lambda = \{\alpha \in \Phi \mid \langle \lambda + \rho, \alpha^\vee \rangle \equiv 0 \bmod l\}. \tag{3.1.1}$$

Thus, because of our restrictions on l when working with U_ζ, etc., we can always take (3.1.1) as the definition of Φ_λ. It is useful to observe that, for any $w \in \widetilde{W}_l$,

$$\overline{w}(\Phi_\lambda) = \Phi_{w\cdot\lambda}.$$

Lemma 3.1.1. *Let l be an odd integer. Assume that l is good for Φ. For $\lambda \in X$, there exists a base Π' for Φ such that $\Pi' \cap \Phi_\lambda$ is a base for Φ_λ. In particular, there exists a $w \in W$ and a subset $J \subseteq \Pi$ such that $w(\Phi_\lambda) = \Phi_J$.*

Proof. This result follows immediately from [Bo, Prop. 24, IV, §7] since Q/Q_λ is l-torsion free, where Q_λ is the root lattice of Φ_λ. □

The lemma may fail if l is not good; e. g., let Φ have type F_4 with $l = 3$. Using [Bo, Appendix, Plate VIII] and taking $\lambda = 0$, we see that Φ_0 has base

$$\{\epsilon_1 - \epsilon_2, \epsilon_2 + \epsilon_4\} \cup \{-\epsilon_3, \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4)\},$$

and so has type $A_2 \times A_2$, and there is clearly no $J \subset \Pi$ for which Φ_J has type $A_2 \times A_2$. It is sometimes useful to use the well-known algorithm of Dynkin, which states that the closed subsystems of Φ can be found by removing one or more nodes from the affine diagram of Φ, then repeating this process on a connected component of the resulting diagram. More generally (for type F_4), if 3 divides l, say $l = 3l'$, then the lemma fails for any $\lambda = (l'-1)\rho$.

A parabolic subgroup $P_J = L_J \ltimes U_J \supseteq B$ of the semisimple group G determines a Richardson orbit $\mathcal{C}_J$ in the nullcone $\mathcal{N} = \mathcal{N}(\mathfrak{g})$ of $\mathfrak{g}$. Namely, P_J has a unique dense orbit in the Lie algebra $\mathfrak{u}_J$ of U_J, and $\mathcal{C}_J$ is the G-orbit of any member of this P_J-orbit. If J, K are W-conjugate subsets of Π, then the Johnston-Richardson theorem implies that $\mathcal{C}_J = \mathcal{C}_K$. Hence, given $\lambda \in X$, if there exists $w \in W$ and $J \subseteq \Pi$ with $w(\Phi_\lambda) = \Phi_J$, then λ defines a Richardson class $\mathcal{C}_\lambda$ in $\mathcal{N}$ by setting $\mathcal{C}_\lambda = \mathcal{C}_J$. We recall that for any parabolic subgroup

P_J, the corresponding Richardson class has Zariski closure $\overline{\mathcal{C}_J} = G\cdot \mathfrak{u}_J$. For $\lambda \in X$ with $w(\Phi_\lambda) = \Phi_J$ as above, we set $\mathcal{N}(\Phi_\lambda) := G\cdot \mathfrak{u}_J \subseteq \mathcal{N}$.

3.2. Given an integer l as in Assumption 1.3.1, of particular interest to us will be

$$\Phi_0 = \Phi_{0,l} = \{\alpha \in \Phi : \ \langle \rho, \alpha^\vee\rangle \equiv 0 \bmod l\}.$$

In the simply laced case, computing Φ_0 entails computing the roots where l divides the height. For $\lambda = 0$, Lemma 3.1.1 in fact holds for this weaker condition on l, as will be discussed in the following sections, and culminating in Theorem 3.5.1.

3.3. Classical Lie algebras. In [CLNP, §3.1-3.7], explicit determinations were given of a $J \subset \Pi$ so that $w(\Phi_0) = \Phi_J$ when $l = p$ is a good prime. The proofs work equally well when l satisfies Assumption 1.3.1. We use the root notation and ordering of Bourbaki [Bo].

Theorem 3.3.1. *Let l be as in Assumption 1.3.1, $\mathfrak{g}$ be a classical simple Lie algebra with Φ of type A_n (resp. B_n), and $h = n+1$ (resp. $2n$) be the Coxeter number of Φ.*

(a) *If $l \geq h$ then $\mathcal{N}(\Phi_0) = \mathcal{N}(\mathfrak{g})$ and $\dim \mathcal{N}(\Phi_0) = |\Phi|$.*

(b) *Suppose that $l < h$ where $h-1 = lm+s$ with $m > 0$ with and $0 \leq s \leq l-1$. Then $\mathcal{N}(\Phi_0) = G\cdot \mathfrak{u}_J$ where $J \subseteq \Pi$ such that when*

(i) *Φ is of type A_n,*

$$\Phi_0 \cong \Phi_J \cong \underbrace{A_m \times \cdots \times A_m}_{s+1 \text{ times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{l-s-1 \text{ times}};$$

where $\dim \mathcal{N}(\Phi_0) = n(n+1) - m(lm+2s-l+2)$.

(ii) *Φ is of type B_n,*

$$\Phi_0 \cong \Phi_J \cong \begin{cases} \underbrace{A_m \times \cdots \times A_m}_{\frac{s}{2} \text{ times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{\frac{l-s-1}{2} \text{ times}} \times B_{\frac{m+1}{2}} & \text{if } s \text{ is even } (m \text{ odd}),\\ \underbrace{A_m \times \cdots \times A_m}_{\frac{s+1}{2} \text{ times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{\frac{l-s-2}{2} \text{ times}} \times B_{\frac{m}{2}} & \text{if } s \text{ is odd } (m \text{ even}). \end{cases}$$

Also,

$$\dim \mathcal{N}(\Phi_0) = \begin{cases} 2n^2 - \frac{m(lm+2s-l+3)+1}{2} & \text{if } s \text{ is even } (m \text{ odd}),\\ 2n^2 - \frac{m(lm+2s-l+3)}{2} & \text{if } s \text{ is odd } (m \text{ even}). \end{cases}$$

Theorem 3.3.2. *Let l be as in Assumption 1.3.1, $\mathfrak{g}$ be a classical simple Lie algebra with Φ of type C_n (resp. D_n), and $h = 2n$ (resp. $2n-2$) be the Coxeter number of Φ.*

(a) *If $l \geq h$ then $\mathcal{N}(\Phi_0) = \mathcal{N}(\mathfrak{g})$ and $\dim \mathcal{N}(\Phi_0) = |\Phi|$.*

(b) *Suppose that $l < h$ where $h+1 = lm+s$ with $m > 0$ and $0 \leq s \leq l-1$. Then $\mathcal{N}(\Phi_0) = G\cdot \mathfrak{u}_J$ where $J \subseteq \Pi$ such that when*

(i) *Φ is of type C_n,*

$$\Phi_0 \cong \Phi_J \cong \begin{cases} \underbrace{A_m \times \cdots \times A_m}_{\frac{s}{2} \text{ times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{\frac{l-s-1}{2} \text{ times}} \times C_{\frac{m-1}{2}} & \text{if } s \text{ is even},\\ \underbrace{A_m \times \cdots \times A_m}_{\frac{s-1}{2} \text{ times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{\frac{l-s}{2} \text{ times}} \times C_{\frac{m}{2}} & \text{if } s \text{ is odd}. \end{cases}$$

Also,

$$\dim\mathcal{N}(\Phi_0)=\begin{cases}2n^2-\frac{m(lm+2s-l-1)+1}{2} & \text{if } s \text{ is even } (m \text{ odd}),\\ 2n^2-\frac{m(lm+2s-l-1)}{2} & \text{if } s \text{ is odd } (m \text{ even}).\end{cases}$$

(iv) Φ *is of type* D_n,

$$\Phi_0\cong\Phi_J\cong\begin{cases}\underbrace{A_m\times\cdots\times A_m}_{\frac{s}{2}\text{ times}}\times\underbrace{A_{m-1}\times\cdots\times A_{m-1}}_{\frac{l-s-1}{2}\text{ times}}\times D_{\frac{m+1}{2}} & \text{if } s \text{ is even and } m\geq 3,\\ \underbrace{A_m\times\cdots\times A_m}_{\frac{s-1}{2}\text{ times}}\times\underbrace{A_{m-1}\times\cdots\times A_{m-1}}_{\frac{l-s}{2}\text{ times}}\times D_{\frac{m+2}{2}} & \text{if } s \text{ is odd},\\ \underbrace{A_m\times\cdots\times A_m}_{\frac{s}{2}\text{ times}}\times\underbrace{A_{m-1}\times\cdots\times A_{m-1}}_{\frac{l-s-1}{2}\text{ times}} & \text{if } s \text{ is even and } m=1.\end{cases}$$

Also,

$$\dim\mathcal{N}(\Phi_0)=\begin{cases}2n^2-2n-\frac{m(lm+2s-l+1)-1}{2} & \text{if } s \text{ is even } (m \text{ odd}),\\ 2n^2-2n-\frac{m(lm+2s-l+1)}{2} & \text{if } s \text{ is odd } (m \text{ even}).\end{cases}$$

3.4. Exceptional Lie algebras. For the exceptional Lie algebras, the computation of Φ_0 and $\mathcal{N}(\Phi_0)$ can be carried out by hand. Note that $\dim\mathcal{N}(\Phi_0)=|\Phi|-|\Phi_0|$ and $\mathcal{N}(\Phi_0)$ is the closure of a Richardson orbit. In most cases this information is enough to determine the orbit (cf. [CLNP, §4.2]). When this information is not sufficient, one can pin down the correct orbit by using the Weyl group (cf. [CLNP, §4.3]). In fact, for computational purposes in Section 4, for each value of l satisfying Assumption 1.3.1, we identify an explicit element $w\in W$ and subset $J\subset\Pi$ such that $w(\Phi_0)=\Phi_J$. Note that the choice of w and J is not unique in general. The computer package MAGMA [BC, BCP] was used to verify this. The tables providing the description of $\mathcal{N}(\Phi_0)$, w, and J are presented in Section 9.1.

3.5. As a consequence of the aforementioned results, we have the following:

Theorem 3.5.1. *Let l be as in Assumption 1.3.1. Then there exists $w\in W$ and a subset $J\subseteq\Pi$ such that $w(\Phi_{0,l})=\Phi_J$.*

3.6. Normality of orbit closures. In this subsection, we consider certain nilpotent orbit closures for a complex semisimple lie algebra $\mathfrak{g}=\mathfrak{g}_{\mathbb{C}}$ having root system Φ, etc. Let G be the complex semisimple, simply connected algebraic group of the same root type as $\mathfrak{g}$. There has been considerable work in determining the normal G-orbit closures in the nilpotent variety $\mathcal{N}=\mathcal{N}(\mathfrak{g})$. Now fix an integer l satisfying Assumption 1.3.1. Let $\Phi_0=\Phi_{0,l}$. We will next verify that in almost all types $\mathcal{N}(\Phi_0)$ is normal. In general the regular, subregular and minimal orbits have normal orbit closures. When Φ is of type A_n then all orbit closures are normal.

In the case when Φ is of type B_n, (resp. C_n, D_n), set $N=2n+1$ (resp. $2n$, $2n$). For $X=A$, (resp. B, C, D), let $\mathcal{P}_X(N)$ be the set of partitions of N parametrizing the set of nilpotent orbits for X_N. A precise description of $\mathcal{P}_X(N)$ is given in [CM, Thm. 5.1.2-5.1.4]. If $\lambda\in\mathcal{P}_X(N)$, let $\mathcal{O}_\lambda$ be the corresponding nilpotent X-orbit. In the case of type D, for very even partitions λ, there are two orbits corresponding to the partition λ. We will denote the two orbits by $\mathcal{O}^{I}_{\lambda}$ and $\mathcal{O}^{II}_{\lambda}$.

Write $N = ml + s$ where $0 \leq s \leq l-1$. One can use [UGA1, CLNP] to show that $\mathcal{N}(\Phi_0) = \overline{\mathcal{O}}_{\lambda_X}$ where $\lambda = (l^m, s)$ and λ_X is the X-collapse of λ (see [CM, p. 99]). We will use the procedure given in [KP] to determine whether $\overline{\mathcal{O}}_{\lambda_X}$ is a normal variety. In order to check this, it suffices to look at all codimension two orbits inside of $\overline{\mathcal{O}}_{\lambda_X}$ and determine the singularity types of each one.

In order to determine the possible codimension two orbits, we start with the Hasse diagram in type A_N and use the collapsing procedure (which preserves the ordering in the Hasse diagram). For type A_N, one has the following configuration given in Figure 1. For those values of s for which the partition does not make sense, the orbit is not present. Put

λ	μ_1	μ_2	γ_1	γ_2	γ_3
(l^m, s)	$(l^{m-1}, l-1, s+1)$	$(l^m, s-1, 1)$	$(l^{m-1}, l-2, s+2)$	$(l^{m-1}, l-1, s, 1)$	$(l^m, s-2, 2)$

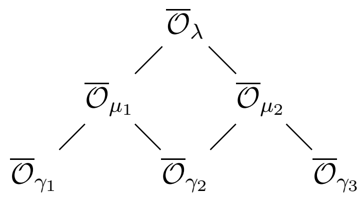

FIGURE 1

If λ_X does not equal $(\mu_1)_X$ or $(\mu_2)_X$, then the possible codimension two orbits in $\overline{\mathcal{O}}_{\lambda_X}$ are $\mathcal{O}_{(\mu_1)_X}$ and $\mathcal{O}_{(\mu_2)_X}$. This occurs when $X = B$ or D and s is odd or when $X = C$ when s is even. In these cases, $\lambda_X = (l^m, s)$ and

$$(\mu_1)_X = \begin{cases} (l^m, 2^2) & X = B, D,\ s = 1,\ l = 3, \\ (l^m, l-2, 3) & X = B, D,\ s = 1,\ l \geq 5, \\ (l^{m-1}, l-2, s+2) & X = B, D,\ 3 \leq s \leq l-4, \\ (l^{m-1}, (l-1)^2) & X = B, D,\ s = l-2, \\ (l^{m-2}, (l-1)^2, s+2) & X = C \end{cases}$$

$$(\mu_2)_X = \begin{cases} (l^m, s-2, 1^2) & X = B, D,\ 3 \leq s \leq l-4, \\ (l^m, l-4, 2) & X = B, D,\ s = l-2,\ l \geq 7, \\ (l^m, 1^3) & X = B, D,\ s = l-2,\ l = 5, \\ (l^m, s-1, 1) & X = C. \end{cases}$$

On the other hand, if $X = B, D$ and s is even ($s \geq 2$) then $\lambda_X = (\mu_2)_X = (l^m, s-1, 1)$. If $s = 0$, we can use the same type of analysis as above. Therefore, the possible codimension two orbits in $\overline{\mathcal{O}}_{\lambda_X}$ are $\mathcal{O}_{(\mu_1)_X}$ and $\mathcal{O}_{(\gamma_3)_X}$. Moreover,

$$(\mu_1)_X = \begin{cases} (l^{m-1}, l-2, s+2) & X = B, D,\ 0 \leq s \leq l-4, \\ (l^{m-1}, (l-2)^2, 1) & X = B, D,\ s = l-3 \end{cases}$$

$$(\gamma_3)_X = \begin{cases} (l^m, s-3, 3) & X = B, D,\ 6 \leq s, \\ (l^m, 2^2) & X = B, D,\ s = 4. \end{cases}$$

The last case occurs when $X = C$ and s is odd. In this situation, $\lambda_C = (\mu_1)_X = (l^{m-1}, l-1, s+1)$. The possible codimension two orbits are $\mathcal{O}_{(\mu_2)_X}$ and $\mathcal{O}_{(\gamma_1)_X}$. The collapse of these partitions is given by

$$(\mu_2)_C = \begin{cases} (l^{m-1}, l-1, s-1, 2) & 3 \leq s, \\ (l^{m-1}, l-1, 1^2) & s = 1 \end{cases}$$

$$(\gamma_1)_C = \begin{cases} (l^{m-1}, l-3, s+3) & s \leq l-6, \\ (l^{m-1}, (l-2)^2) & s = l-4. \end{cases}$$

We will apply the procedure in [KP, §0.8] to verify the normality for $\mathcal{N}(\Phi_0)$. According to [KP, Thm. 1], to show normality it suffices to look at the type of singularities in the classes of codimension two. In order to determine the types of singularities, one can strip off the common first rows and columns of λ_X, and the corresponding partitions give the codimension two classes. By the description above of these partitions, this process results in partitions with less than or equal to three parts. This shows that the singularities are not of "Type e" ([KP, Table 3.4]). Therefore, for the classical Lie algebras, $\mathcal{N}(\Phi_0)$ is a normal variety.

In the exceptional types by using the tables in Section 9.1 and work of Sommers [So1, Thm. 1.1], it follows that $\mathcal{N}(\Phi_0)$ is normal for E_6. For E_7 the techniques used in [So2] can be used to verify the normality of $\mathcal{N}(\Phi_0)$ in those cases [So3]. In the case of E_8, Sommers [So3] was able to verify that $\mathcal{N}(\Phi_0)$ is normal in all cases except when $l = 7, 9$. We believe that normality will still hold for these last two cases, but a proof is unavailable at this time. For this reason we state our main results with care and precision. For F_4, one can can use [Br1, Thm. 1] to show that $\mathcal{N}(\Phi_0)$ is normal, and for G_2 the only orbits which realize $\mathcal{N}(\Phi_0)$ are the regular and subregular orbits which have normal orbit closures.

We summarize the analysis in this subsection in the following theorem.

Theorem 3.6.1. *Let l be as in Assumption 1.3.1 and $J \subseteq \Pi$ so that $\mathcal{N}(\Phi_0) = G \cdot \mathfrak{u}_J$. If Φ is of type E_8, assume that $l \neq 7, 9$. Then $\mathcal{N}(\Phi_0)$ is a normal variety.*

3.7. Resolution of singularities.

We maintain the notation of the previous subsection. Let $J \subseteq \Pi$ and P_J be the associated parabolic subgroup. The moment map $G \times_{P_J} \mathfrak{u}_J \to G \cdot \mathfrak{u}_J$ is a resolution of singularities if $C_G(x) \subseteq P_J$ where x is a Richardson element contained in $\mathfrak{u}_J$. When the moment map is a resolution of singularities and $G \cdot x$ has a normal orbit closure, then

$$\operatorname{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*) \cong \mathbb{C}[G \times_{P_J} \mathfrak{u}_J] \cong \mathbb{C}[G \cdot \mathfrak{u}_J]$$

(e.g. [So1, p. 290] [KP, §0.7]).

Theorem 3.7.1. *Let l be as in Assumption 1.3.1 and $J \subseteq \Pi$ so that $\mathcal{N}(\Phi_0) = G \cdot \mathfrak{u}_J$. If Φ is of type E_8, assume that $l \neq 7, 9$. Then*

$$\operatorname{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*) \cong \mathbb{C}[G \cdot \mathfrak{u}_J].$$

Proof. In Theorem 3.6.1, we showed that $\mathcal{N}(\Phi_0)$ is a normal variety under the hypothesis of the theorem. So it suffices to prove that $C_G(x) \subseteq P_J$ where $x \in \mathfrak{u}_J$. First observe that $C_G(x)^0 \subseteq P_J$ [Car, Cor. 5.2.2] where $C_G(x)^0$ is the identity component of $C_G(x)$. For type

A_n, all orbits are Richardson. The centralizer is generated by the connected component and toral elements, so the centralizer is contained in P_J. So in this case we have verified the theorem. Second, if the Richarson orbit corresponding to x is even, then P_J coincides with the "Bala-Carter" parabolic and in this case we have $C_G(x) \subseteq P_J$ (e.g. [Jan3, p. 93 Rem.]). For the exceptional groups, we can use the description of $\mathcal{N}(\Phi_0)$ given in Section 9.1 and the tables in [Car, p. 401-407] to verify that either the orbit is even or the centralizer is connected.

We are left to verify our claim for the remaining classical groups of types B_n, C_n and D_n. Let $N = 2n+1$ (resp. $2n$, $2n$) for $X = B_n$ (resp. C_n, D_n). Define integers m, s by $N = lm+s$ where $0 \leq s \leq l-1$. Set $\lambda = (l^m, s)$ and recall that $\mathcal{N}(\Phi_0) = \overline{\mathcal{O}}_{\lambda_X}$ where λ_X is the X-collapse of λ.

For type B_n we have

$$\lambda_B = \begin{cases} (l^m, s) & \text{if } s \text{ is odd or } s = 0, \\ (l^m, s-1, 1) & \text{if } s \text{ is even and } s \neq 0. \end{cases}$$

In either case each of the nonzero parts are odd so the associated weighted Dynkin diagram has even entries [CM, §5.3]. Therefore, the orbit $\mathcal{O}_{\lambda_B}$ is even, and the centralizer is contained in P_J. A similar argument works to show that for type D_n the orbit $\mathcal{O}_{\lambda_D}$ is even.

We are left to analyze type C_n. In this case

$$\lambda_C = \begin{cases} (l^m, s) & \text{if } m \text{ is even } (s \text{ even}), \\ (l^{m-1}, l-1, s+1) & \text{if } m \text{ is odd } (s \text{ odd}). \end{cases}$$

In the first case when m is even (s even), let b be the number of distinct even nonzero parts. So $b = 1$ and by [CM, p. 92] the component group $A(\mathcal{O}_{\lambda_C}) \cong (\mathbb{Z}/2\mathbb{Z})^{b-1}$. Therefore, the centralizer is connected in this case.

Now suppose that m is odd (s odd). In this case the component group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ so we need to appeal to a different line of reasoning. In [H, Cor. 7.7], Hesselink provides a necessary and sufficient criterion for having $C_G(x) \subseteq P_J$. Here λ_C satisfies condition (i) of [H, Cor. 7.7] where $\epsilon = 1$ and $\lambda_C \in \text{Pai}(2n, m-1)$ (see [H, §6.1] for the definitions). □

4. Combinatorics with the Steinberg module

One of the key steps in computing the cohomology $\mathrm{H}^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$ for $l > h$ entails showing that the $u_\zeta(\mathfrak{t})$ invariants on $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}), \mathbb{C})$ is one-dimensional by looking at weights of $\Lambda^\bullet_{\zeta,\varnothing}$. This is far from being true for $l \leq h$. For small l, a more intricate analyis is needed, where we consider the multiplicity of a certain "Steinberg module" in $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})$. This computation will then be used in Sections 5 and 6 to make the desired cohomology computations.

4.1. Steinberg weights. If l satisfies Assumption 1.3.1, by Theorem 3.5.1, we can choose $w \in W$ such that $w(\Phi_0) = \Phi_{w\cdot 0} = \Phi_J$.

Lemma 4.1.1. *For all $\alpha \in J$, $\langle w \cdot 0, \alpha^\vee \rangle = l - 1$.*

Proof. Since $w \cdot 0 = w(\rho) - \rho$, the claim is equivalent to showing that $l = \langle w(\rho), \alpha^\vee \rangle = \langle \rho, w^{-1}(\alpha)^\vee \rangle$ for all $\alpha \in J$. But $w^{-1}(J)$ is the unique set Π_0 of simple roots for Φ_0 contained in $\Phi_0^+ = \Phi^+ \cap \Phi_0$. So, the lemma asserts that $\langle \rho, \beta^\vee \rangle = l$ for all $\beta \in \Pi_0$. Therefore, although $w \in W$ is not uniquely determined, if the lemma holds for one choice of w, it holds for all choices. Now, in the exceptional cases, for each l, an element $w \in W$ and a subset $J \subseteq \Pi$ satisfying $w(\Phi_0) = \Phi_J$ are identified in Section 9.1. In these cases, the lemma can be checked directly by using the tables in Section 9.2.

Now assume that Φ has classical type A_n, B_n, C_n, or D_n. We can also assume that $\Pi_0 \neq \emptyset$. Then Φ_0 consists of all roots α such that the coroot $\alpha^\vee$ has height $\mathrm{ht}(\alpha^\vee) = \langle \rho, \alpha^\vee \rangle$ divisible by ℓ. In particular, $\ell < h$. If $\beta^\vee$ is any coroot of height $m > 0$, then, for any positive integer i, $3 \leq i < m$, it is easy to see (in each possible case), that we can write $\beta^\vee = \delta^\vee + \gamma^\vee$ for $\delta, \gamma \in \Phi$ satisfying $\mathrm{ht}(\delta^\vee) = i$. Then $\beta = a\delta + b\gamma$, where a, b are positive rational numbers. If $\beta \in \Phi_0^+$ with $\mathrm{ht}(\beta^\vee) = t\ell$, $t > 1$, we can thus write $\beta^\vee = \delta^\vee + \gamma^\vee$ with $\delta, \gamma \in \Phi_0^+$. If $\beta \in \Pi_0$, it follows that $\mathrm{ht}(\beta^\vee) = \ell$, i. e., $\langle \rho, \beta^\vee \rangle = \ell$, as required. □

Set $M := (\mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^*$. By the lemma, we see that $w \cdot 0$ is a J-Steinberg weight (see Section 2.10). The module M is therefore isomorphic to a "Steinberg" type module on $U_\zeta(\mathfrak{l}_J)$ that remains irreducible if viewed as a $u_\zeta(\mathfrak{l}_J)$-module. The highest weight of M is $-w_{0,J}(w \cdot 0)$ and the lowest weight of M is $-w \cdot 0$.

4.2. Weights of $\Lambda^\bullet_{\zeta,J}$.

By Section 2.7, the $\overline{\mathrm{Ad}}$-action induces an action of $U_\zeta(\mathfrak{p}_J)$ (and hence also of $u_\zeta(\mathfrak{p}_J)$) on $\mathcal{U}_\zeta(\mathfrak{u}_J)$. This defines an action of $U_\zeta(\mathfrak{p}_J)$ on the cohomology $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})$. See also Section 2.8. In Theorem 4.3.1 below, we determine

$$\mathrm{Hom}_{u_\zeta(\mathfrak{l}_J)}((\mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^*, \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})).$$

By Lemma 2.9.1(b), it follows that, as a U_ζ^0-module, $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})$ is a subquotient of $\Lambda^\bullet_{\zeta,J}$. The key ingredient to proving Theorem 4.3.1 is the following computation about weights in $\Lambda^\bullet_{\zeta,J}$.

Proposition 4.2.1. *Let l be as in Assumption 1.3.1. Choose $w \in W$ such that $w(\Phi_0) = \Phi_J$. Let γ be a J-dominant weight of $\Lambda^i_{\zeta,J}$ and write $\gamma = -w_{0,J}(w \cdot 0) + l\nu$ for some $\nu \in X(T)$.*

(a) *Suppose that $l \nmid n+1$ when Φ is of type A_n and $l \neq 9$ when Φ is of type E_6. Then $\gamma = -w_{0,J}(w \cdot 0)$ (i. e., $\nu = 0$) and $i = \ell(w)$.*

(b) *If Φ is of type A_n with $n+1 = l(m+1)$ and w is as defined in (4.8.1), then γ is one of the following, for $0 \leq t \leq l-1$:*

$$\gamma = -w_{0,J}(w \cdot 0) + l\varpi_{t(m+1)} \qquad \text{with} \qquad i = \ell(w) + (m+1)t(l-t).$$

We set $\varpi_0 = 0$.

(c) *If Φ is of type E_6 and $l = 9$ (assuming that w and J are as in Section 9.1), then γ is one of the following:*

$$\gamma = -w_{0,J}(w \cdot 0) \text{ with } i = \ell(w) = 8,$$

$$\gamma = -w_{0,J}(w \cdot 0) + \ell\varpi_1 \text{ with } i = 20,$$

$$\gamma = -w_{0,J}(w \cdot 0) + \ell\varpi_6 \text{ with } i = 20.$$

One should observe that the weight ν in the statement of the proposition must necessarily be J-dominant by Lemma 4.1.1. The proposition will be proved below. See Section 4.4.

4.3. Multiplicity of the Steinberg module. Assuming that Proposition 4.2.1 holds, we can now determine how often the "Steinberg module" M (introduced in Section 4.1) appears in $\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}_J),\mathbb{C})$. When $l \geq h$, $w = 1$, $J = \varnothing$, and $M = \mathbb{C}$.

Theorem 4.3.1. *Let l be as in Assumption 1.3.1. Choose $w \in W$ such that $w(\Phi_0) = \Phi_J$.*

(a) *Suppose that $l \nmid n+1$ when Φ is of type A_n and $l \neq 9$ when Φ is of type E_6. Then as $U_\zeta(\mathfrak{l}_J)$-modules*

$$\mathrm{Hom}_{u_\zeta(\mathfrak{l}_J)}((\mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0)^*, \mathrm{H}^i(\mathcal{U}_\zeta(\mathfrak{u}_J),\mathbb{C})) = \begin{cases} \mathbb{C} & \text{if } i = \ell(w) \\ 0 & \text{otherwise.} \end{cases}$$

(b) *If Φ is of type A_n with $n+1 = l(m+1)$ and w is as defined in (4.8.1), then as $U_\zeta(\mathfrak{l}_J)$-modules*

$$\begin{aligned}\mathrm{Hom}_{u_\zeta(\mathfrak{l}_J)}&((\mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0)^*, \mathrm{H}^i(\mathcal{U}_\zeta(\mathfrak{u}_J),\mathbb{C})) \\ &= \begin{cases} \mathbb{C} & \text{if } i = \ell(w) \\ l\varpi_{t(m+1)} \oplus l\varpi_{(l-t)(m+1)} & \text{if } i = \ell(w) + (m+1)t(l-t) \\ & \quad \text{for } 1 \leq t \leq (l-1)/2 \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

(c) *If Φ is of type E_6 and $l = 9$ (assuming that w and J are as in Section 9.1), then as $U_\zeta(\mathfrak{l}_J)$-modules*

$$\mathrm{Hom}_{u_\zeta(\mathfrak{l}_J)}((\mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0)^*, \mathrm{H}^i(\mathcal{U}_\zeta(\mathfrak{u}_J),\mathbb{C})) = \begin{cases} l\varpi_1 \oplus l\varpi_6 & \text{if } i = 20 \\ \mathbb{C} & \text{if } i = \ell(w) = 8 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since $\mathrm{Hom}_{u_\zeta(\mathfrak{l}_J)}((\mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0)^*, \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J),\mathbb{C}))$ is a module for $U_\zeta(\mathfrak{l}_J)$ on which $u_\zeta(\mathfrak{l}_J)$ acts trivially, it is also a (finite dimensional—hence completely reducible) module for the universal enveloping algebra $\mathbb{U}(\mathfrak{l}_J)$ (see Section 2.3). Thus, if $\mathrm{Hom}_{u_\zeta(\mathfrak{l}_J)}((\mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0)^*, \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J),\mathbb{C})) \neq 0$, any $U_\zeta(\mathfrak{l}_J)$-composition factor will be of the form $L_J(\nu)^{[1]}$ for a J-dominant weight ν. In other words, there must be a nonzero $U_\zeta(\mathfrak{l}_J)$-homomorphism

$$(\mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0)^* \otimes L_J(\nu)^{[1]} \to \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J),\mathbb{C}). \tag{4.3.1}$$

Hence, the weight $-w_{0,J}(w\cdot 0) + l\nu$ must appear in $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J),\mathbb{C})$, and so also in $\Lambda^\bullet_{\zeta,j}$ by Lemma 2.9.1(b). The theorem now follows from Proposition 4.2.1 if each weight listed therein does indeed give rise to a non-trivial homomorphism as in (4.3.1).

By Lemma 2.9.1(a),

$$\sum_{n=0}^{\infty}(-1)^n \,\mathrm{ch}\,\mathrm{H}^n(\mathcal{U}_\zeta(\mathfrak{u}_J),\mathbb{C}) = \sum_{n=0}^{\dim(\mathfrak{u}_J)} (-1)^n \,\mathrm{ch}\,\Lambda^n_{\zeta,J}. \tag{4.3.2}$$

Since $\mathrm{H}^{\bullet}(\mathbb{U}(\mathfrak{u}_J),\mathbb{C})$ can be computed from $\Lambda^{\bullet}(\mathfrak{u}_J^*)$ considered as a complex (with appropriately defined differential), we similarly have

$$\sum_{n=0}^{\dim(\mathfrak{u}_J)}(-1)^n\,\mathrm{ch}\,\mathrm{H}^n(\mathbb{U}(\mathfrak{u}_J),\mathbb{C})=\sum_{n=0}^{\dim(\mathfrak{u}_J)}(-1)^n\,\mathrm{ch}\,\Lambda^n(\mathfrak{u}_J^*). \tag{4.3.3}$$

Clearly, $\mathrm{ch}_\zeta\,\Lambda^n_{\zeta,J}=\mathrm{ch}\,\Lambda^n(\mathfrak{u}_J^*)$. Hence (4.3.2) and (4.3.3) give

$$\sum_{n=0}^{\infty}(-1)^n\,\mathrm{ch}\,\mathrm{H}^n(\mathcal{U}_\zeta(\mathfrak{u}_J),\mathbb{C})=\sum_{n=0}^{\dim(\mathfrak{u}_J)}(-1)^n\,\mathrm{ch}\,\mathrm{H}^n(\mathbb{U}(\mathfrak{u}_J),\mathbb{C}). \tag{4.3.4}$$

Let ${}^JW=\{x\in W\mid x(\Phi^-)\cap\Phi^+\subset\Phi^+\backslash\Phi_J^+\}$ be the set of distinguished right W_J-coset representatives in W. By [W, Thm. 2.5.1.3],

$$\sum_{n=0}^{\dim(\mathfrak{u}_J)}(-1)^n\,\mathrm{ch}\,\mathrm{H}^n(\mathbb{U}(\mathfrak{u}_J),\mathbb{C})=\sum_{x\in{}^JW}(-1)^{\ell(x)}\mathrm{ch}\,L_J(-w_{0,J}(x\cdot 0)). \tag{4.3.5}$$

Also, (cf. [HK, Prop. 3.4.5])

$$\mathrm{ch}_\zeta\,\mathrm{ind}_{U_\zeta(\mathfrak{b}_{\mathfrak{l}_J})}^{U_\zeta(\mathfrak{l}_J)}(-w_{0,J}(x\cdot 0))=\mathrm{ch}\,L_J(-w_{0,J}(x\cdot 0)).$$

Combining this with (4.3.4) and (4.3.5) gives

$$\sum_{n=0}^{\infty}(-1)^n\,\mathrm{ch}\,\mathrm{H}^n(\mathcal{U}_\zeta(\mathfrak{u}_J),\mathbb{C})=\sum_{x\in{}^JW}(-1)^{\ell(x)}\,\mathrm{ch}\,\mathrm{ind}_{U_\zeta(\mathfrak{b}_{\mathfrak{l}_J})}^{U_\zeta(\mathfrak{l}_J)}(-w_{0,J}(x\cdot 0)). \tag{4.3.6}$$

The weights γ in Proposition 4.2.1 are all "Steinberg weights" whose induced module is injective over $U_\zeta(\mathfrak{l}_J)$. Hence, they do not appear as a composition factor in any other induced module. For part (a), since only one such weight occurs, it gives rise to a composition factor on the right-hand side of (4.3.6) which cannot be canceled out by any other factor. Hence it appears as well on the left-hand side which completes the proof. For parts (b) and (c), while multiple "Steinberg weights" appear, these weights are all distinct and hence give rise to distinct composition factors on the right-hand side of (4.3.6) which cannot cancel each other out. Hence they all appear on the left-hand side as well. □

Remark 4.3.2. In those cases where the cohomology is two-dimensional, the weights are not strongly linked (cf. [Jan1, II.6.4]). Hence the isomorphisms in the theorem also hold as $U_\zeta(\mathfrak{p}_J)$-modules.

4.4. Proof of Proposition 4.2.1. The remainder of Section 4 is devoted to proving Proposition 4.2.1. Note first of all that the weight $-w_{0,J}(w\cdot 0)$ does appear in $\Lambda^{\bullet}_{\zeta,J}$ in degree $\ell(w)$ (cf. [GW, 7.3], [FP1, Prop. 2.2]). So the goal is to show that (in most cases) a weight ν satisfying the hypothesis must in fact be zero. In Sections 4.5–4.6, the classical root systems will be considered. For these, we will mainly work with the ϵ-basis that represents Φ [Bo, p. 250] and $\langle -,-\rangle$ will always denote the ordinary Euclidean inner product. In Section 4.5, we first show that $\langle\nu,\alpha^\vee\rangle=0$ for all $\alpha\in J$. To do this, for each of the classical root systems

$X \in \{A_n, B_n, C_n, D_n\}$ we show the existence of an element $\delta_X \in X(T)$ with the following properties:

$$\max_{\lambda}\langle \lambda, \delta_X\rangle := \max\{\langle \lambda, \delta_X\rangle \mid \lambda \text{ a weight of } \Lambda^{\bullet}_{\zeta,J}\} = \langle -w_{0,J}(w\cdot 0), \delta_X\rangle \text{ and} \tag{4.4.1}$$

$$\langle \varpi_j, \delta_X\rangle > 0 \text{ for all fundamental weights } \varpi_j \text{ corresponding to } \alpha_j \in J. \tag{4.4.2}$$

When Φ is of type A_n or C_n, one can choose $\delta_X = \sum_{\alpha\in J}\alpha^\vee$ and the maximum value in (4.4.1) turns out to be simply $(l-1)|J|$.

Now assume that a δ_X satisfying properties (4.4.1) and (4.4.2) exists and that $-w_{0,J}(w\cdot 0) + l\nu$ is a J-dominant weight of $\Lambda^{i}_{\zeta,J}$ for some J-dominant weight ν. Then

$$\langle -w_{0,J}(w\cdot 0), \delta_X\rangle + l\langle \nu, \delta_X\rangle = \langle -w_{0,J}(w\cdot 0) + l\nu, \delta_X\rangle \le \max_{\lambda}\langle \lambda, \delta_X\rangle = \langle -w_{0,J}(w\cdot 0), \delta_X\rangle.$$

This forces ν to vanish on J. To show that $\nu = 0$, it remains to show that $\langle \nu, \alpha^\vee\rangle = 0$ for $\alpha \in \Pi\backslash J$. In Section 4.6, this is shown for types B_n, C_n, D_n. Sections 4.7 and 4.8 are devoted to dealing with type A_n and showing how the "extra" weights arise in Proposition 4.2.1.

In Section 4.9, similar ideas along with direct computations will be used to deal with the exceptional root systems.

4.5. The weight $\boldsymbol{\delta_X}$. In this section, we construct a weight δ_X which satisfies properties (4.4.1) and (4.4.2). We will first consider the case $m \ge 2$ in Theorems 3.3.1 and 3.3.2.

For Φ of type A_n and

$$\Phi_0 \cong \Phi_J \cong \underbrace{A_m \times \cdots \times A_m}_{r_1 \text{ times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{r_2 \text{ times}};$$

let

$$\delta_A = (\underbrace{1, 0, \dots, 0-1, \dots, 1, 0, \dots, 0, -1}_{r_1 \text{ times}}, \underbrace{1, 0, \dots, 0, -1, \dots, 1, 0, \dots, 0, -1}_{r_2 \text{ times}})$$

in the orthonormal basis describing Φ in $n+1$-dimensional Euclidean space $\mathbb{E}$ [Bo, p. 250]. Note that the first r_1-groupings of $(1, 0, \dots, 0, -1)$ have $m+1$-components, while the last r_2-groupings of $(1, 0, \dots, 0, -1)$ have m-components. In this case, $\delta_A = \sum_{\alpha\in J}\alpha^\vee$, and evidently $\langle \varpi_j, \delta_A\rangle > 0$ for all ϖ_j corresponding to simple roots in J (property (4.4.2)).

To show property (4.4.1), let λ be a weight of $\Lambda^{i}_{\zeta,J}$. Then λ is a sum of distinct positive roots not in Φ^+_J. If β is a positive root then $\langle \beta, \delta_A\rangle = 0, \pm 1, \pm 2$ (i. e. $\beta = \epsilon_i - \epsilon_j$ with $i < j$). Set

$$A[t] = \{\beta \in \Phi^+\backslash\Phi^+_J \mid \langle \beta, \delta_A\rangle = t\}.$$

A quick count shows that

$$\begin{aligned} |A[2]| &= \frac{(r_1+r_2-1)(r_1+r_2)}{2} \\ |A[1]| &= (m-1)(r_1+r_2-1)r_1 + (m-2)(r_1+r_2-1)r_2. \end{aligned}$$

Therefore,

$$\begin{aligned} \max\nolimits_{\lambda}\langle \lambda, \delta_A\rangle &= 2|A[2]| + |A[1]| \\ &= (r_1+r_2-1)(mr_1 + (m-1)r_2) \\ &= (r_1+r_2-1)|J|. \end{aligned}$$

According to Theorem 3.3.1, we can set $r_1 = s+1$ and $r_2 = l-s-1$. Consequently,

$$\max_\lambda \langle \lambda, \delta_A \rangle = (l-1)|J| = \langle -w_{0,J}(w\cdot 0), \delta_A \rangle$$

because A_n is simply laced.

For the other classical Lie algebras, let Φ be of type X_n where $X = B, C, D$ with

$$\Phi_0 \cong \Phi_J \cong \underbrace{A_m \times \cdots \times A_m}_{r_1 \text{ times}} \times \underbrace{A_{m-1} \times \ldots A_{m-1}}_{r_2 \text{ times}} \times X_q;$$

with

$$\delta_X = (\underbrace{1,0,\ldots,0-1,\ldots,1,0,\ldots,0,-1}_{r_1 \text{ times}}, \underbrace{1,0,\ldots,0,-1,\ldots,1,0,\ldots,0,-1}_{r_2 \text{ times}}, 1,0,\ldots,0).$$

Notice that in all cases $-w_{0,J}\delta_X = \delta_X$. By using [GW, p. 102], one can verify that $\langle \varpi_j, \delta_X \rangle > 0$ for all ϖ_j that correspond to simple roots in J. If β is a positive root in X_n then $\langle \beta, \delta_X \rangle = 0, \pm 1, \pm 2$, so set $X[t] = \{\beta \in \Phi^+ \backslash \Phi_J^+ \mid \langle \beta, \delta_X \rangle = t\}$. By using our computations for type A_n with consideration of other positive roots in X_n, it follows that

$$|X[2]| = \begin{cases} (r_1+r_2)^2 & X_n = B_n; \\ (r_1+r_2)(r_1+r_2+1) & X_n = C_n; \\ (r_1+r_2)^2 & X_n = D_n \end{cases}$$

and

$$|X[1]| = \begin{cases} 2(r_1+r_2)(r_1(m-1)+r_2(m-2)) + (2q-1)(r_1+r_2) & X_n = B_n; \\ 2(r_1+r_2)(r_1(m-1)+r_2(m-2)) + 2(q-1)(r_1+r_2) & X_n = C_n; \\ 2(r_1+r_2)(r_1(m-1)+r_2(m-2)) + 2(q-1)(r_1+r_2) & X_n = D_n. \end{cases}$$

Hence,

$$\begin{aligned} \max_\lambda \langle \lambda, \delta_X \rangle &= 2|X[2]| + |X[1]| \\ &= \begin{cases} 2(r_1+r_2)(r_1 m + r_2(m-1)) + (2q-1)(r_1+r_2) & X_n = B_n; \\ 2(r_1+r_2)(r_1 m + r_2(m-1)) + 2q(r_1+r_2) & X_n = C_n; \\ 2(r_1+r_2)(r_1 m + r_2(m-1)) + 2(q-1)(r_1+r_2) & X_n = D_n. \end{cases} \end{aligned}$$

On the other hand, by using the expression of ρ for the classical Lie algebras in terms of the ϵ-basis (see [GW, p. 107]), one sees that

$$\langle -w_{0,J}(w\cdot 0), \delta_X \rangle = \begin{cases} (l-1)(r_1 m + r_2(m-1)) + (l-1)(q-\frac{1}{2}) & X_n = B_n; \\ (l-1)(r_1 m + r_2(m-1)) + (l-1)(q) & X_n = C_n; \\ (l-1)(r_1 m + r_2(m-1)) + (l-1)(q-1) & X_n = D_n. \end{cases}$$

In Theorems 3.3.1 and 3.3.2, $r_1 + r_2 = \frac{l-1}{2}$ for all root systems Φ of types B_n, C_n, or D_n, and so $\max_\lambda \langle \lambda, \delta_X \rangle = \langle -w_{0,J}(w\cdot 0), \delta_X \rangle$ as desired.

Next we consider the case $m = 1$. Again let Φ be of type X_n where $X = A, B, C, D$ with

$$\Phi_0 \cong \Phi_J \cong \begin{cases} \underbrace{A_1 \times \cdots \times A_1}_{r \text{ times}} & \text{for } X_n = A_n,\ C_n, \text{ or } D_n. \\ \underbrace{A_1 \times \cdots \times A_1}_{r \text{ times}} \times A_1 & \text{for } X_n = B_n. \end{cases}$$

and

$$\delta_X = \begin{cases} (\underbrace{1, -1, \dots, 1, -1}_{r \text{ times}}, \underbrace{0, \dots, 0}_{z \text{ times}}) & \text{for } X_n = A_n,\ C_n, \text{ or } D_n; \\ (\underbrace{1, -1, \dots, 1, -1}_{r \text{ times}}, \underbrace{0, \dots, 0}_{z \text{ times}}, 1) & \text{for } X_n = B_n. \end{cases}$$

As above, one can verify that $\langle \varpi_j, \delta_X \rangle > 0$ for all ϖ_j that correspond to simple roots in J. Also, we conclude that

$$|X[2]| = \begin{cases} \frac{r(r-1)}{2} & X_n = A_n; \\ r^2 & X_n = B_n,\ C_n; \\ r(r-1) & X_n = D_n. \end{cases}$$

and

$$|X[1]| = \begin{cases} rz & X_n = A_n; \\ 2rz & X_n = C_n,\ D_n; \\ 2rz + r + z & X_n = B_n. \end{cases}$$

Hence,

$$\max\nolimits_\lambda \langle \lambda, \delta_X \rangle \quad = \quad 2|X[2]| + |X[1]| = \begin{cases} r(r + z - 1) & X_n = A_n; \\ (r + \frac{1}{2})(2r + 2z) & X_n = B_n; \\ r(2r + 2z) & X_n = C_n; \\ r(2(r + z - 1)) & X_n = D_n. \end{cases}$$

On the other hand, by using the expression of ρ for the classical Lie algebras in terms of the ϵ-basis (see [GW, p. 107]) one sees that

$$\langle -w_{0,J}(w \cdot 0), \delta_X \rangle = \begin{cases} r(l-1) & X_n = A_n,\ C_n,\ D_n; \\ (r + \frac{1}{2})(l-1) & X_n = B_n. \end{cases}$$

By Theorems 3.3.1 and 3.3.2, r is the number of copies of A_1 and $z = n + 1 - 2r$ (resp. $n - 2r - 1$, $n - 2r$) in type A_n (resp. B_n, C_n or D_n). One obtains that

$$r + z = \begin{cases} l & X_n = A_n; \\ \frac{l-1}{2} & X_n = B_n,\ C_n; \\ \frac{l+1}{2} & X_n = D_n. \end{cases}$$

Again, we see that $\max_\lambda \langle \lambda, \delta_X \rangle = \langle -w_{0,J}(w \cdot 0), \delta_X \rangle$.

4.6. Types $\boldsymbol{B_n, C_n, D_n}$. In this subsection Φ is always of type B_n, C_n or D_n. Under this assumption we show that the only weight ν satisfying the hypothesis of Proposition 4.2.1 is the zero weight. Note that our restriction on the root systems and l being odd implies that $\gcd(l, (X:Q)) = 1$. For any such weight ν we observe that $l\nu \in Q$ because $-w_{0,J}(w \cdot 0) + l\nu$ is a weight of $\Lambda^i_{\zeta,J}$. It follows that $\nu \in Q$.

Our results in Section 4.5 show that both $-w_{0,J}(w \cdot 0)$ and $-w_{0,J}(w \cdot 0) + l\nu$ consist of a sum of all the roots in $X[1] \cup X[2]$ with some additional terms involving roots in $X[0]$. Therefore, $l\nu$ is a sum of distinct roots in $X[0] \cup -X[0]$.

We express δ_X and ν in the ϵ-basis as $\delta_X = \sum_{i=1}^n \delta_{X,i}\epsilon_i$ and $\nu = \sum_{i=1}^n \nu_i\epsilon_i$, respectively. Notice that $\nu \in Q$ implies that $\nu \in \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_n$. Define the following sets

$$S_{(a,b)} = \{(i,j) : \delta_{X,i} = a,\ \delta_{X,j} = b, \text{ and } i < j\} \text{ and } S_{(a)} = \{i :\ \delta_{X,i} = a\}. \tag{4.6.1}$$

The case when Φ is of type B_n will be discussed in detail. The verification that $\nu = 0$ for the cases when Φ is of type C_n and D_n are left to the reader.

Any positive root of the form $\epsilon_i - \epsilon_j$ in $B[0]$ satisfies $(i,j) \in S_{(0,0)} \cup S_{(1,1)} \cup S_{(-1,-1)}$, any positive root of the form $\epsilon_i + \epsilon_j$ in $B[0]$ satisfies $(i,j) \in S_{(0,0)} \cup S_{(1,-1)} \cup S_{(-1,1)}$ and any positive root of the form ϵ_i in $B[0]$ satisfies $i \in S_{(0)}$. Using (4.6.1), we can express

$$\begin{aligned} l\nu &= \sum_{(i,j)\in S_{(1,1)}} m_{i,j}(\epsilon_i - \epsilon_j) + \sum_{(i,j)\in S_{(-1,-1)}} \widetilde{m}_{i,j}(\epsilon_i - \epsilon_j) \\ &+ \sum_{(i,j)\in S_{(1,-1)}} n_{i,j}(\epsilon_i + \epsilon_j) + \sum_{(i,j)\in S_{(-1,1)}} \widetilde{n}_{i,j}(\epsilon_i + \epsilon_j) \\ &+ \sum_{(i,j)\in S_{(0,0)}} p_{i,j}(\epsilon_i - \epsilon_j) + \sum_{(i,j)\in S_{(0,0)}} \widetilde{p}_{i,j}(\epsilon_i + \epsilon_j) + \sum_{i\in S_{(0)}} q_i\epsilon_i \end{aligned}$$

with $m_{i,j}, \widetilde{m}_{i,j}, n_{i,j}, \widetilde{n}_{i,j}, q_i, p_{i,j}, \widetilde{p}_{i,j} = 0, \pm 1$.

The above expression shows that if $\delta_{B,i} = 1$ then $l\nu_i = \sum_j (m_{i,j} + n_{i,j} + m_{j,i} + \widetilde{n}_{j,i})$. For fixed i, it follows from Theorem 3.3.1 that the number of pairs of the form (i,j) together with those of form (j,i) in $S_{(1,1)}$ are less than $(l+1)/2$. A similar counting argument shows that the number of pairs of the form (i,j) in $S_{(1,-1)}$ together with the number of pairs of the form (j,i) in $S_{(-1,1)}$ are also less than $(l-1)/2$. Therefore, $|\sum_j (m_{i,j} + n_{i,j} + m_{j,i} + \widetilde{n}_{j,i})| \le (l-1)$ and $\delta_{B,i} = 1$ implies $\nu_i = 0$. Similarly, one can argue that $\delta_{B,i} = -1$ implies $\nu_i = 0$.

Any simple root $\epsilon_i - \epsilon_{i+1} \in \Pi\backslash J$ satisfies either $\delta_{B,i} = -1$ and $\delta_{B,i+1} = 1$ or $\delta_{B,i} = 0$ and $\delta_{B,i+1} = 0$. It follows from above that, for any $\alpha \in \Pi\backslash J$,

$$\langle l\nu, \alpha^\vee \rangle = \langle \sum_{i\in S_{(0)}} q_i\epsilon_i + \sum_{(i,j)\in S_{(0,0)}} p_{i,j}(\epsilon_i - \epsilon_j) + \sum_{(i,j)\in S_{(0,0)}} \widetilde{p}_{i,j}(\epsilon_i + \epsilon_j), \alpha^\vee \rangle.$$

For $m > 1$, there are no roots in $\Pi\backslash J$ with $\delta_{B,i} = 0$ and $\delta_{B,i+1} = 0$ and the inner product on the right-hand side vanishes. Since ν vanishes on J, one concludes $\nu = 0$, as desired.

Assume $m = 1$ and let i be such that $\delta_{B,i} = 0$, then $l\nu_i = q_i + \sum_j (p_{i,j} + \tilde{p}_{i,j} + p_{j,i} + \tilde{p}_{j,i})$. Theorem 3.3.1 implies that for fixed i there are less than $(l-1)/2$ pairs of the form (i,j) or (j,i) in $S_{(0,0)}$. It follows that $l|\nu_i| = |q_i + \sum_j (p_{i,j} + \tilde{p}_{i,j} + p_{j,i} + \tilde{p}_{j,i})| < l$. This forces $\nu = 0$.

4.7. Type A_n. In this (and the next) subsection Φ is always of type A_n with simple roots $\alpha_1, \ldots, \alpha_n$. We will show that a weight ν satisfying the hypothesis of Proposition 4.2.1 equals the zero weight unless l divides $n+1$. For $\mu \in Q$, μ_i always denotes the coefficient of μ in its expansion in terms of the ϵ-basis (i. e., $\mu = \sum_{i=1}^{n+1} \mu_i \epsilon_i$).

First consider the case when $m > 1$ in Theorem 3.3.1. Let

$$\Phi_0 \cong \Phi_J \cong \underbrace{A_m \times \cdots \times A_m}_{s+1 \text{ times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{l-s-1 \text{ times}}$$

where $n = lm + s$ and $0 \leq s \leq l-1$.

We start out by choosing a particular $w \in W$ with $w(\Phi_0) = \Phi_J$. Partition the set $\{\frac{n}{2}, \frac{n}{2} - 1, ..., -\frac{n}{2} + 1, -\frac{n}{2}\}$, i. e., the set of coordinates of ρ in the ϵ-basis, into its congruence classes modulo l and order each congruence class in decreasing order. Then we order the congruence classes according to the size of their largest element from highest to lowest. The resulting array is the coordinate vector of a W-conjugate of ρ. We denote this conjugate by $w\rho$ and $w \in W$ denotes the unique permutation that sends ρ to $w\rho$. If we identify the Weyl group with the symmetric group in $n+1$ letters, then w can be described as follows.

$$\text{For } 1 \leq i \leq n+1 \text{ define } s_i, t_i \text{ via } i - 1 = s_i l + t_i \text{ with } 0 \leq t_i \leq l-1. \tag{4.7.1}$$

$$\text{Then } w(\epsilon_i) = \epsilon_{w(i)} \text{ where } w(i) = \begin{cases} t_i(m+1) + s_i + 1 & \text{if } 0 \leq t_i \leq s \\ t_i m + s_i + s + 2 & \text{if } s+1 \leq t_i \leq l-1. \end{cases} \tag{4.7.2}$$

$$\text{Moreover, } w_{0,J} w(i) = \begin{cases} (t_i+1)(m+1) - s_i & \text{if } 0 \leq t_i \leq s \\ (t_i+1)m - s_i + s + 1 & \text{if } s+1 \leq t_i \leq l-1. \end{cases} \tag{4.7.3}$$

For any $u \in W$ define $Q(u) := \{\alpha \in \Phi^+ \mid u\alpha \in \Phi^-\}$. Using (4.7.1) and (4.7.2) we find that [GW, 7.3]

$$Q(w) = \{\epsilon_i - \epsilon_j \mid s_i < s_j, t_i > t_j\} \text{ and } w \cdot 0 = \sum_{\{\gamma \in Q(w)\}} -w(\gamma). \tag{4.7.4}$$

Define

$$f(i) = \begin{cases} w_{0,J} w(lm + i) = 1 + (i-1)(m+1) & \text{if } 1 \leq i \leq s+1 \\ w_{0,J} w(l(m-1) + i) = 1 + (i-1)m + (s+1) & \text{if } s+2 \leq i \leq l, \end{cases}$$

and

$$g(i) = \begin{cases} w_{0,J} w(i) = i(m+1) & \text{if } 1 \leq i \leq s+1 \\ w_{0,J} w(i) = im + (s+1) & \text{if } s+2 \leq i \leq l. \end{cases} \tag{4.7.5}$$

Next set $\delta^i = \epsilon_{f(i)} - \epsilon_{g(i)}$ for $1 \leq i \leq l$. Then $\delta_A = \sum_{i=1}^{l} \delta^i = \sum_{i=1}^{l} \epsilon_{f(i)} - \epsilon_{g(i)}$.
The $\alpha_{g(i)} = \epsilon_{g(i)} - \epsilon_{f(i+1)}$ are precisely the simple roots contained in Π but not in J.

Using the notation introduced in (4.6.1), we partition $A[0]$ into the following subsets:

$$\begin{aligned} R^+_{(0,0)} &= \{\epsilon_i - \epsilon_j \in A[0] \mid (i,j) \in S_{(0,0)}\}, \\ R^+_{(1,1)} &= \{\epsilon_i - \epsilon_j \in A[0] \mid (i,j) \in S_{(1,1)}\} = \{\epsilon_{f(i)} - \epsilon_{f(j)} \mid 1 \le i < j \le l\}, \\ R^+_{(-1,-1)} &= \{\epsilon_i - \epsilon_j \in A[0] \mid (i,j) \in S_{(-1,-1)}\} = \{\epsilon_{g(i)} - \epsilon_{g(j)} \mid 1 \le i < j \le l\}. \end{aligned}$$

In addition we set $R_{(a,a)} = R^+_{(a,a)} \cup -R^+_{(a,a)}, a \in \{-1,0,1\}$. Notice that both sets $R_{(1,1)}$ and $R_{(-1,-1)}$ form root systems of type A_{l-1} with simple roots

$$\beta_i := \epsilon_{f(i)} - \epsilon_{f(i+1)} \text{ and } \tau_i := \epsilon_{g(i)} - \epsilon_{g(i+1)}, \tag{4.7.6}$$

respectively.

Next define

$$S^+ := \{\epsilon_{f(i)} - \epsilon_{f(j)} \mid 1 \le i \le s+1, s+2 \le j \le l\} \text{ and } S := S^+ \cup -S^+.$$

Then it follows from (4.7.3) through (4.7.5) that

$$R^+_{(-1,-1)} \cap w_{0,J}w(Q(w)) = \varnothing \text{ and } R^+_{(1,1)} \cap w_{0,J}w(Q(w)) = S^+. \tag{4.7.7}$$

One concludes that the weight $-w_{0,J}(w \cdot 0)$ is the sum of all roots in $A[1] \cup A[2]$ together with certain roots in $R^+_{(0,0)}$ and the roots in S^+. The elements of S^+ can also be characterized as those roots in $R^+_{(1,1)}$ that contain β_{s+1}. It is important to note that no roots of $R^+_{(-1,-1)}$ contribute to $-w_{0,J}(w \cdot 0)$.

Next assume that λ is a weight of $\Lambda^\bullet_{\zeta,J}$ such that $\langle \lambda + w_{0,J}(w \cdot 0), \alpha \rangle = 0$ for all $\alpha \in J$. Set $\nu = \lambda + w_{0,J}(w \cdot 0)$. Using the β-basis of $R_{(1,1)}$ and the γ-basis of $R_{(-1,-1)}$, we express ν in the form

$$\nu = \sum_{i=1}^{l-1} k_i \beta_i + \sum_{i=1}^{l-1} l_i \tau_i + \sum_{\eta \in R^+_{(0,0)}} m_\eta \eta. \tag{4.7.8}$$

Since ν is the zero weight when restricted to J, one observes that $\langle \nu, \delta^i \rangle = 0$ for $1 \le i \le l$. Since $\langle \nu, \delta^1 \rangle = 0$, $k_1 - l_1 = 0$ and, inductively, it follows from $\langle \nu, \delta^i \rangle = 0$ that $k_i = l_i$ for $1 \le i \le l-1$. Moreover, it follows from (4.7.7) that all k_i and l_i are nonnegative. One concludes that ν is a sum of distinct roots in $R_{(0,0)}$ together with distinct roots in $R^+_{(-1,-1)}$ and in $R^+_{(1,1)} \backslash S^+$.

Finally, assume that λ_1 and λ_2 are two weights of $\Lambda^\bullet_{\zeta,J}$ such that $\langle \lambda_i + w_{0,J}(w \cdot 0), \alpha \rangle = 0$ for all $\alpha \in J$. For example λ_1 could be of the form $-w_{0,J}(u \cdot 0)$ for some $u \ne w$ with $u(\Phi_0) = \Phi_J$ and λ_2 could be equal to $-w_{0,J}(u \cdot 0) + l\nu$, where ν is the zero weight when restricted to J. It follows from our above arguments that $\lambda_2 - \lambda_1$ is a sum of distinct roots in $R_{(-1,-1)} \cup R_{(1,1)} \backslash S \cup R_{(0,0)}$. The elements in $R_{(1,1)} \backslash S$ form a root system of type $A_s \times A_{l-s-2}$, spanned by the roots $\{\beta_1, \ldots, \beta_s\} \cup \{\beta_{s+2}, \ldots, \beta_{l-1}\}$, as defined in (4.7.6). We can decompose $\lambda_2 - \lambda_1 = \gamma_1 + \gamma_2 + \gamma_3$ where the support of γ_1 lies entirely in the type A_s component of $R_{(1,1)} \backslash S$, the support of γ_2 lies entirely in the type A_{l-s-2} component of $R_{(1,1)} \backslash S$, and the support of γ_3 lies entirely in $R_{(-1,-1)} \cup R_{(0,0)}$. Next observe that $\alpha_{g(i)} = \beta_i - \delta^i$. It follows that $\langle \lambda_2 - \lambda_1, \alpha_{g(i)} \rangle = \langle \lambda_2 - \lambda_1, \beta_i \rangle = \langle \gamma_1 + \gamma_2, \beta_i \rangle$. The inner product of $\lambda_2 - \lambda_1$ with the roots in $\Pi \backslash J$ is then given by the following:

$$\langle \lambda_2 - \lambda_1, \alpha_{g(i)} \rangle = \begin{cases} \langle \gamma_1, \beta_i \rangle & \text{if } 1 \leq i \leq s, \\ \langle \gamma_1 + \gamma_2, \beta_i \rangle & \text{if } i = s+1, \\ \langle \gamma_2, \beta_i \rangle & \text{if } s+2 \leq i \leq l-1. \end{cases} \tag{4.7.9}$$

Since γ_1 is a sum of distinct roots of a root system of type A_s a direct computation shows that $|\langle \gamma_1, \beta_i \rangle| \leq s+1$. Similarly, $|\langle \gamma_2, \beta_i \rangle| \leq l-s-1$. Now $\lambda_2 - \lambda_1 \in lX$ implies that either $\lambda_1 = \lambda_2$ or $l = s+1$. Hence, Proposition 4.2.1 holds for type A_n as long as l does not divide $n+1$.

Suppose now that $m = 1$. Then

$$\Phi_0 \cong \Phi_J \cong \underbrace{A_1 \times \cdots \times A_1}_{s+1 \text{ times}}.$$

Essentially the same argument as above works here as well. We highlight some of the differences and leave the details to the interested reader. The definition of w holds as above with $m = 1$. Define $f(i)$, $g(i)$, δ^i, δ_A, and $\alpha_{g(i)}$ just as above with $m = 1$. Note that for $i \geq s+2$, $f(i) = g(i)$ and $\delta^i = 0$. In the definitions of $R^+_{(1,1)}$ and $R^+_{(-1,-1)}$, we have $1 \leq i < j \leq s+1$, and the root systems $R_{(1,1)}$ and $R_{(-1,-1)}$ are of type A_s, while $S^+ = \varnothing$. For $s+2 \leq i \leq l-1$, the $\beta_i = \tau_i$ form a basis for the root system $R_{(0,0)}$ of type A_{l-s-2}. The equivalent of (4.7.7) is now

$$R^+_{(a,a)} \cap w_{0,J} w(Q(w)) = \varnothing \text{ where } a \in \{1, 0, -1\}.$$

In (4.7.8), the index i should run from $i = 1$ to $i = s$. Next we consider $\lambda_2 - \lambda_1$, as defined above. We decompose $\lambda_2 - \lambda_1 = \gamma_1 + \gamma_2 + \gamma_3$ with the support of γ_1 in $R_{(1,1)}$, the support of γ_2 in $R_{(0,0)}$ and the support of γ_3 in $R_{(-1,-1)}$. Then equation (4.7.9) remains valid. Hence one obtains the same conclusion.

4.8. Type A_n with l dividing $n+1$.

We continue to identify the Weyl group with the symmetric group in $n+1$ letters. Assume throughout this subsection that l divides $n+1$. Then $l = s+1$ and the definition of w given in (4.7.1) and (4.7.2) can be simplified to

$$w(\epsilon_i) = \epsilon_{w(i)} \text{ where } w(i) = t_i(m+1) + s_i + 1. \tag{4.8.1}$$

We now follow the arguments used in [AJ, 6.2]. Recall that $n+1 = (m+1)l$. We define the element $\sigma \in W$ as follows:

$$\sigma = (1, 2, ..., l)(l+1, l+2, ..., 2l) \cdots (ml+1, ml+2, ..., (m+1)l).$$

Direct computation shows that $w(\sigma^t \cdot 0) = -l\varpi_{g(t)}$ for $1 \leq t \leq l-1$. Setting $\varpi_{g(0)} = 0$ yields

$$w \cdot 0 = w\sigma^t \cdot 0 + l\varpi_{g(t)} \text{ for } 0 \leq t \leq l-1. \tag{4.8.2}$$

We find that

$$\begin{aligned} Q(w\sigma^t) = \{\epsilon_i - \epsilon_j \mid s_i < s_j, \sigma^t(t_i+1) > \sigma^t(t_j+1)\} \\ \cup \{\epsilon_i - \epsilon_j \mid s_i = s_j, t_i < t_j \text{ and } \sigma^t(t_i+1) > \sigma^t(t_j+1)\}. \end{aligned}$$

The cardinality of the first set in the above union is equal to the cardinality of $Q(w)$ (see (4.7.4)) while the second set can be identified with $Q(\sigma^t)$. Using this decomposition of $Q(w\sigma^t)$, [GW, 7.3], and [AJ, 6.2(3)], we conclude that

$$\ell(w\sigma^t) = \ell(w) + \ell(\sigma^t) = \ell(w) + (m+1)t(l-t). \tag{4.8.3}$$

Next, assume that $l\nu - w_{0,J}(w\cdot 0)$ is a weight of $\Lambda^{\bullet}_{\zeta,J}$ such that $\langle l\nu, \alpha\rangle = 0$ for all $\alpha \in J$. The discussion in Section 4.7 shows that $l\nu$ is a sum of distinct roots in $R^+_{(-1,-1)} \cup R_{(0,0)} \cup R^+_{(1,1)}$. We can decompose $\nu = \gamma_1 + \gamma_2$ where the support of γ_1 lies entirely in $R^+_{(1,1)}$ and the support of γ_2 lies entirely in $R^+_{(-1,-1)} \cup R_{(0,0)}$. As before, the inner product $\langle l\nu, \alpha_{g(i)}\rangle = \langle l\nu, \beta_i\rangle = \langle l\gamma_1, \beta_i\rangle$ is completely determined by the contribution coming from $R^+_{(1,1)}$, the positive roots of a type A_{l-1} root system. By [AJ, 2.2, 6.1] one concludes that either $\gamma_1 = 0$ or $l\gamma_1 = lx\kappa_i$, where κ_i denotes the fundamental weight corresponding to the simple root β_i of the root system of type A_{l-1} and x is in the corresponding Weyl group. But γ_1 is a sum of positive roots. Hence $lx\kappa_i$ must also be a sum of positive roots. But this can only happen if $lx\kappa_i = l\kappa_i$. Therefore, we must have $l\nu = l\varpi_{g(i)}$. Finally, by (4.8.2), for each $1 \le i \le l-1$, we have

$$-w_{0,J}(w\cdot 0) + l\varpi_{g(i)} = -w_{0,J}(w\sigma^i \cdot 0),$$

and the latter weight is a weight of $\Lambda^{\bullet}_{\zeta,J}$ (cf. [GW, 7.3],[FP1, Prop. 2.2]). Further, by (4.8.3), this lies in degree $\ell(w\sigma^i) = \ell(w) + (m+1)t(l-i)$. Since $g(i) = i(m+1)$, the result follows.

4.9. Exceptional Lie algebras: In this subsection, we assume that the root system Φ is of exceptional type. We show that if ν satisfies the hypothesis of Proposition 4.2.1, then $\nu = 0$ except in the case when Φ is of type E_6 and $l = 9$. Note that in the excluded case l is divisible by $(X : \mathbb{Z}\Phi) = 3$. Our goal is to show that if $-w_{0,J}(w\cdot 0) + l\nu$ with ν being J-dominant is a weight of $\Lambda^{\bullet}_{\zeta,J}$, then $\nu = 0$. For the exceptional Lie algebras, an explicit choice of w and J is listed in Section 9.1. One can then explicitly compute the value of $-w_{0,J}(w\cdot 0)$. This was again done with the aid of MAGMA [BC, BCP] and the results are given in tables in Section 9.2. The weights in the tables are listed with respect to the basis $\{\varpi_1, \dots, \varpi_n\}$ of fundamental dominant weights.

Since the dimension of $\Lambda^{\bullet}_{\zeta,J}$ is finite, with the aid of a computer, one could in principle compute all possible weights of $\Lambda^{\bullet}_{\zeta,J}$ and compare them to $-w_{0,J}(w\cdot 0)$ modulo l. For types F_4 and G_2, this can readily be done and one finds that $\nu = 0$. For type E_n, the size of $\Lambda^{\bullet}_{\zeta,J}$ is sufficiently large as to make the computations somewhat impractical on a typical desktop computer. As such, we present an alternative approach which makes use of some of the ideas from the preceding sections on classical root systems to show directly that $\nu = 0$ or reduce the computations to a more manageable number.

In what follows, let $\delta^\vee = \sum_{\alpha\in J}\alpha^\vee$. This is a slight abuse of notation since $\delta^\vee$ may not equal $(\sum_{\alpha\in J}\alpha)^\vee$ but should not lead to any confusion. Recall that $\langle -w_{0,J}(w\cdot 0), \delta^\vee\rangle = (l-1)|J|$. As in Section 4.2.1, set $\max_\lambda\langle\lambda, \delta^\vee\rangle := \max\{\langle\lambda, \delta^\vee\rangle \mid \lambda \text{ a weight of } \Lambda^{i}_{\zeta,J}\}$. As with the classical cases, the key to showing that $\nu = 0$ is that $\max_\lambda\langle\lambda, \delta^\vee\rangle$ is in general "close" to $(l-1)|J|$.

To make this more precise, set

$$E[t] := \{\beta \in \Phi^+ \backslash \Phi^+_J \mid \langle\beta, \delta^\vee\rangle = t\}.$$

Then we can decompose $\Phi^{+}\backslash\Phi_{J}^{+}$ as a disjoint union:

$$\Phi^{+}\backslash\Phi_{J}^{+} = E[<0] \cup E[0] \cup E[>0]$$

where

$$E[>0] = \cup_{t>0} E[t] \text{ and } E[<0] = \cup_{t<0} E[t].$$

That is, we separate the positive roots into those which give a positive, zero, or negative inner product with $\delta^{\vee}$. Since weights in $\Lambda^{\bullet}_{\zeta,J}$ are composed of sums of distinct positive roots from $\Phi^{+}\backslash\Phi_{J}^{+}$, clearly,

$$\max_{\lambda}\langle\lambda,\delta^{\vee}\rangle = \langle \sum_{\beta\in E[>0]} \beta, \delta^{\vee}\rangle.$$

For convenience, for the remainder of this section, set $\lambda := \sum_{\beta\in E[>0]}\beta$. One might think of λ as a conical representative of those weights having maximum inner product with $\delta^{\vee}$. Any other such element would be of the form $\lambda + z$ where z is a sum of distinct roots which lie in $E[0]$. With the aid of MAGMA the weight λ can be readily computed for a given l, w, and J. This is given in the tables in Section 9.2.

Let $x = -w_{0,J}(w\cdot 0) + l\nu$ be a J-dominant weight of $\Lambda^{\bullet}_{\zeta,J}$. To show that $\nu = 0$, we show that $\langle\nu,\alpha^{\vee}\rangle = 0$ for all $\alpha\in\Pi$. We separate this into two cases: $\alpha\in J$ and $\alpha\in\Pi\backslash J$. The first case follows if it can be shown that

$$|\langle -w_{0,J}(w\cdot 0),\delta^{\vee}\rangle - \langle\lambda,\delta^{\vee}\rangle| < l.$$

This can be done by direct calculation (see tables in Section 9.2) and holds in all but one case (type E_8 with $l = 7$, which is dealt with separately).

We now outline the basic process for showing the second case. Once the first case is known, that is, $\langle\nu,\alpha^{\vee}\rangle = 0$ for $\alpha\in J$, we have that $\langle x,\delta^{\vee}\rangle = \langle -w_{0,J}(w\cdot 0) + l\nu,\delta^{\vee}\rangle = \langle -w_{0,J}(w\cdot 0),\delta^{\vee}\rangle = (l-1)|J|$. To show that $\langle\nu,\alpha^{\vee}\rangle = 0$ for $\alpha\in\Pi\backslash J$, we express x in the form $x = \lambda - a + b + z$ where a, b, and z consist of sums of distinct roots lying in $E[>0]$, $E[<0]$, and $E[0]$, respectively. While z can consist of arbitrarily many elements from $E[0]$, a and b are constrained to consist of a small number of elements from $E[>0]$ or $E[<0]$ depending upon how close $\langle\lambda,\delta^{\vee}\rangle$ is to $(l-1)|J|$. In each case, this can be explicitly described.

Given $\alpha\in\Pi\backslash J$, $\langle -w_{0,J}(w\cdot 0),\alpha^{\vee}\rangle$ and $\langle\lambda,\alpha^{\vee}\rangle$ can be explicitly computed. Also, one can find bounds A and B (integers) such that for an arbitirary linear combination z as above, one has $A \le \langle z,\alpha^{\vee}\rangle \le B$. Using the expression of x as $\lambda + a - b + z$, and considering the possibilities for a and b, one can then obtain bounds $A' \le \langle x,\alpha^{\vee}\rangle \le B'$. In those cases needed, such bounds are listed in Section 9.2. On the other hand,

$$\langle x,\alpha^{\vee}\rangle = \langle -w_{0,J}(w\cdot 0) + l\nu,\delta^{\vee}\rangle = \langle -w_{0,J}(w\cdot 0),\alpha^{\vee}\rangle + l\langle\nu\alpha^{\vee}\rangle.$$

Comparing this to the bounds obtained previously, one often finds that $\langle\nu,\alpha^{\vee}\rangle$ must be zero. When the bounds allow for a nonzero ν, MAGMA is used to verify that no solutions exist (except in type E_6 when $l = 9$) by checking all possibilities for a, b, and z. When needed for efficiency, the known bounds can be used to limit the possible choices for z. The basic details for each case are given below.

Type E_6:

$l = 11$: Here $\lambda = -w_{0,J}(w \cdot 0)$. Hence $x = \lambda + z$ (i. e., a and b must be empty), and z would need to equal $l\nu$. However, for $\alpha \in \Pi\backslash J$, one finds that $-4 \leq \langle z, \alpha^\vee \rangle \leq 6$. Hence, z cannot equal 11ν unless $\nu = 0$.

$l = 9$: In this case, $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$ and $\langle \lambda, \delta^\vee \rangle$ differ by two. Also $E[> 0] = E[1]$ and $E[< 0] = E[-1]$. So there are three ways we can express x in the form $\lambda - a + b + z$:

(1) $x = \lambda - x_1 - x_2 + z$ where $x_1, x_2 \in E[1]$ $(x_1 \neq x_2)$,
(2) $x = \lambda - x_1 + y_1 + z$ where $x_1 \in E[1]$ and $y_1 \in E[-1]$,
(3) $x = \lambda + y_1 + y_2 + z$ where $y_1, y_2 \in E[-1]$ $(y_1 \neq y_2)$.

Using MAGMA, we compute all possibilities. In case (1), we find precisely one pair of elements in $E[1]$ that works with z being an empty sum. That is, $-w_{0,J}(w \cdot 0)$ is a sum of eight distinct roots in $E[1]$. In case (2), no sums over $E[0]$ work. In case (3), we find however two cases where λ plus two elements of $E[-1]$ and eight elements of $E[0]$ equals $-w_{0,J}(w \cdot 0) + 9\nu$ for a J-dominant weight ν. In one case $\nu = \varpi_1$ and in the other $\nu = \varpi_6$. The reader should be aware that these weights give rise to the exceptions stated in Theorem 1.3.2(b)(iii).

$l = 7$: Here $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle = \langle \lambda, \delta^\vee \rangle$. Hence x must be of the form $x = \lambda + z$. For $\alpha \in \Pi\backslash J$, one finds the bounds listed in Section 9.2. For each α, in order to have $x = -w_{0,J}(w \cdot 0) + 7\nu$, we must have $\langle \nu, \alpha^\vee \rangle = 0$.

$l = 5$: In this case, $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$ and $\langle \lambda, \delta^\vee \rangle$ differ by one. So there are two ways we can express x in the form $\lambda - a + b + z$:

(1) $x = \lambda - x_1 + z$ where $x_1 \in E[1]$,
(2) $x = \lambda + y_1 + z$ where $y_1 \in E[-1]$.

Using MAGMA, we compute all possibilities and find that only one such x works - in case (1) with z being the sum of a pair of elements from $E[0]$. Again, $x = -w_{0,J}(w \cdot 0)$, i. e., $\nu = 0$.

Type E_7:

$l = 17$: Here $\lambda = -w_{0,J}(w \cdot 0)$. Hence $x = \lambda + z$, and z would need to equal $l\nu$. However, for $\alpha \in \Pi\backslash J$, one finds that $-9 \leq \langle z, \alpha^\vee \rangle \leq 10$. Hence, z cannot equal 17ν unless $\nu = 0$.

$l = 15$: In this case, $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$ and $\langle \lambda, \delta^\vee \rangle$ differ by two. So there are three ways we can express x in the form $\lambda - a + b + z$ just as in the type E_6, $l = 9$ case. For $\alpha \in \Pi\backslash J$, one finds the bounds listed in Section 9.2. For each α, in order to have $x = -w_{0,J}(w \cdot 0) + 15\nu$, we must have $\langle \nu, \alpha^\vee \rangle = 0$.

$l = 13$: Here $\lambda = -w_{0,J}(w \cdot 0)$. Hence $x = \lambda + z$, and z would need to equal $l\nu$. However, for $\alpha \in \Pi\backslash J$, one finds that $-6 \leq \langle z, \alpha^\vee \rangle \leq 8$. Hence, z cannot equal 13ν unless $\nu = 0$.

$l = 11$: Here $\lambda = -w_{0,J}(w \cdot 0)$. Hence $x = \lambda + z$, and z would need to equal $l\nu$. However, for $\alpha \in \Pi\backslash J$, one finds that $-4 \leq \langle z, \alpha^\vee \rangle \leq 6$. Hence, z cannot equal 11ν unless $\nu = 0$.

$l = 9$: In this case, $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$ and $\langle \lambda, \delta^\vee \rangle$ differ by two. Here, for all $\beta \in E[> 0]$, $\langle \beta, \delta^\vee \rangle \in \{1, 2, 3\}$, and for all $\beta \in E[< 0]$, $\langle \beta, \delta^\vee \rangle \in \{-1, -2, -3\}$. So there are five ways we can express x in the form $\lambda - a + b + z$. We leave the details to the interested reader.

For $\alpha \in \Pi\backslash J$, one finds the bounds listed in Section 9.2. For each α, in order to have $x = -w_{0,J}(w \cdot 0) + 9\nu$, we must have $\langle \nu, \alpha^\vee \rangle = 0$.

$l = 7$: In this case, $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$ and $\langle \lambda, \delta^\vee \rangle$ differ by three. Here, for all $\beta \in E[>0]$, $\langle \beta, \delta^\vee \rangle \in \{1, 2, 3\}$, and for all $\beta \in E[<0]$, $\langle \beta, \delta^\vee \rangle \in \{-1, -2, -3\}$. So there are ten ways we can express x in the form $\lambda - a + b + z$. In this case, for $\alpha \in \Pi\backslash J$, the bounds on $\langle x, \alpha^\vee \rangle$ allow for the possibility that $\langle \nu, \alpha^\vee \rangle \neq 0$. Using MAGMA, we compute all possibilities. We find precisely one case that works (of form $x = \lambda - a$ where a is a sum of three distinct roots in $E[1]$ and b and z are empty) with $x = -w_{0,J}(w \cdot 0)$, i. e., $\nu = 0$.

$l = 5$: In this case, $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$ and $\langle \lambda, \delta^\vee \rangle$ differ by three. As in the $l = 7$ case, there are ten ways we can express x in the form $\lambda - a + b + z$. Here $\Pi\backslash J = \{\alpha_4\}$ and one finds that $-10 \leq \langle x, \alpha_4^\vee \rangle \leq -6$. Since $\langle -w_{0,J}(w \cdot 0), \alpha_4^\vee \rangle = -9$, in order to have $x = -w_{0,J}(w \cdot 0) + 5\nu$, we must have $\langle \nu, \alpha_4^\vee \rangle = 0$.

Type E_8:

$l = 29$: Here $\lambda = -w_{0,J}(w \cdot 0)$. Hence $x = \lambda + z$, and z would need to equal $l\nu$. However, for $\alpha \in \Pi\backslash J$, one finds that $-16 \leq \langle z, \alpha^\vee \rangle \leq 18$. Hence, z cannot equal 29ν unless $\nu = 0$.

$l = 27$: In this case, $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$ and $\langle \lambda, \delta^\vee \rangle$ differ by two. There are three ways we can express x in the form $\lambda - a + b + z$. For $\alpha \in \Pi\backslash J$ one finds the bounds listed in Section 9.2. In order to have $x = -w_{0,J}(w \cdot 0) + 27\nu$, we must have $\langle \nu, \alpha^\vee \rangle = 0$.

$l = 25$: In this case, $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$ and $\langle \lambda, \delta^\vee \rangle$ differ by four. There are five ways we can express x in the form $\lambda - a + b + z$. For $\alpha \in \Pi\backslash J$ one finds the bounds listed in Section 9.2. In order to have $x = -w_{0,J}(w \cdot 0) + 25\nu$, we must have $\langle \nu, \alpha^\vee \rangle = 0$.

$l = 23$: In this case, $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle = \langle \lambda, \delta^\vee \rangle$. Hence $x = \lambda + z$. For $\alpha \in \Pi\backslash J$ one finds the bounds listed in Section 9.2. In order to have $x = -w_{0,J}(w \cdot 0) + 23\nu$, we must have $\langle \nu, \alpha^\vee \rangle = 0$.

$l = 21$: In this case, $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$ and $\langle \lambda, \delta^\vee \rangle$ differ by four. Here, for all $\beta \in E[>0]$, $\langle \beta, \delta^\vee \rangle \in \{1, 2\}$, and for all $\beta \in E[<0]$, $\langle \beta, \delta^\vee \rangle \in \{-1, -2\}$. So there are fourteen ways we can express x in the form $\lambda - a + b + z$. For $\alpha \in \Pi\backslash J$ one finds the bounds listed in Section 9.2. In order to have $x = -w_{0,J}(w \cdot 0) + 21\nu$, we must have $\langle \nu, \alpha^\vee \rangle = 0$.

$l = 19$: In this case, $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle = \langle \lambda, \delta^\vee \rangle$. Hence $x = \lambda + z$. For $\alpha \in \Pi\backslash J$ one finds the bounds listed in Section 9.2. In order to have $x = -w_{0,J}(w \cdot 0) + 19\nu$, we must have $\langle \nu, \alpha^\vee \rangle = 0$.

$l = 17$: In this case, $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle = \langle \lambda, \delta^\vee \rangle$. Hence $x = \lambda + z$. For $\alpha \in \Pi\backslash J$ one finds the bounds listed in Section 9.2. In order to have $x = -w_{0,J}(w \cdot 0) + 17\nu$, we must have $\langle \nu, \alpha^\vee \rangle = 0$.

$l = 15$: In this case, $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$ and $\langle \lambda, \delta^\vee \rangle$ differ by eight. Here, for all $\beta \in E[>0]$, $\langle \beta, \delta^\vee \rangle \in \{1, 2, 3\}$, and for all $\beta \in E[<0]$, $\langle \beta, \delta^\vee \rangle \in \{-1, -2, -3\}$. As a result, there are numerous ways we can express x in the form $\lambda - a + b + z$. Further, for $\alpha \in \Pi\backslash J$, the bounds on $\langle x, \alpha^\vee \rangle$ allow for the possibility that $\langle \nu, \alpha^\vee \rangle \neq 0$. By analyzing the constraints placed on

a, b, and z (to afford a nonzero ν), the resulting possiblities are all computed with MAGMA, and one finds that in fact no nonzero solution exists.

$l=13$: In this case, $\langle -w_{0,J}(w\cdot 0),\delta^{\vee}\rangle$ and $\langle \lambda,\delta^{\vee}\rangle$ differ by three. There are ten ways we can express x in the form $\lambda-a+b+z$. For $\alpha\in\Pi\backslash J$ one finds the bounds listed in Section 9.2. In order to have $x=-w_{0,J}(w\cdot 0)+13\nu$, we must have $\langle \nu,\alpha^{\vee}\rangle=0$.

$l=11$: In this case, $\langle -w_{0,J}(w\cdot 0),\delta^{\vee}\rangle$ and $\langle \lambda,\delta^{\vee}\rangle$ differ by two. There are five ways we can express x in the form $\lambda-a+b+z$. For $\alpha\in\Pi\backslash J$ one finds the bounds listed in Section 9.2. In order to have $x=-w_{0,J}(w\cdot 0)+11\nu$, we must have $\langle \nu,\alpha^{\vee}\rangle=0$.

$l=9$: In this case, $\langle -w_{0,J}(w\cdot 0),\delta^{\vee}\rangle$ and $\langle \lambda,\delta^{\vee}\rangle$ differ by three. There are ten ways we can express x in the form $\lambda-a+b+z$. For $\alpha\in\Pi\backslash J$, the bounds on $\langle x,\alpha^{\vee}\rangle$ allow for the possibility that $\nu\neq 0$. To reduce the number of possibilities that need to be checked, observe that there is a sizable difference between $\langle \lambda,\alpha_1^{\vee}\rangle=18$ and $\langle -w_{0,J}(w\cdot 0),\alpha_1^{\vee}\rangle=8$. Since $\langle \nu,\alpha_1^{\vee}\rangle=0$, we must have $\langle -a+b+z,\alpha_1^{\vee}\rangle=-10$. We find that $\langle -a+b,\alpha_1^{\vee}\rangle\geq -3$ and $\langle z,\alpha_1^{\vee}\rangle\geq -8$. This reduces the possibilities to a number manageable for MAGMA to compute all the possible cases. The only x that works is precisely $-w_{0,J}(w\cdot 0)$, i. e., $\nu=0$.

$l=7$: In this case, $\langle -w_{0,J}(w\cdot 0),\delta^{\vee}\rangle$ and $\langle \lambda,\delta^{\vee}\rangle$ differ by 8 which is larger than 7. Here we may not immediately conclude that $\langle \nu,\alpha^{\vee}\rangle=0$ for all $\alpha\in J$. However, since $\langle \nu,\alpha^{\vee}\rangle\geq 0$ for all $\alpha\in J$, this must be true for all but possibly one α for which one could have $\langle \nu,\alpha^{\vee}\rangle=1$.

Suppose the latter case holds, then we would have $\langle x,\delta^{\vee}\rangle=49$ whereas $\langle \lambda,\delta^{\vee}\rangle=50$. So there would be only two ways in which x could occur ($x=\lambda-x_1+z$ where $x_1\in E[1]$ or $x=\lambda+y_1+z$ where $y_1\in E[-1]$). Here $\Pi\backslash J=\{\alpha_4\}$ and one finds that $-21\leq\langle x,\alpha_4^{\vee}\rangle\leq -19$. On the other hand, $\langle -w_{0,J}(w\cdot 0),\alpha_4^{\vee}\rangle=-17$. Since $\langle \nu,\alpha_4^{\vee}\rangle$ is an integer, these bounds show that we cannot have $x=-w_{0,J}(w\cdot 0)+7\nu$ for any ν, contradicting our assumption. Therefore $\langle \nu,\alpha^{\vee}\rangle=0$ for all $\alpha\in J$.

Now, our standard argument can be used to show that $\langle \nu,\alpha_4^{\vee}\rangle=0$. Since $\langle -w_{0,J}(w\cdot 0),\delta^{\vee}\rangle$ and $\langle \lambda,\delta^{\vee}\rangle$ differ by 8 there are numerous ways we can express x in the form $\lambda-a+b+z$. One finds that $-21\leq\langle x,\alpha^{\vee}\rangle\leq -12$. Since $\langle -w_{0,J}(w\cdot 0),\alpha_4^{\vee}\rangle=-17$, In order to have $x=-w_{0,J}(w\cdot 0)+7\nu$, we must have $\langle \nu,\alpha_4^{\vee}\rangle=0$.

Type F_4:

$l=11$: Here $\lambda=-w_{0,J}(w\cdot 0)$. Hence $x=\lambda+z$, and z would need to equal $l\nu$. However, for $\alpha\in\Pi\backslash J$, one finds that $-4\leq\langle z,\alpha^{\vee}\rangle\leq 5$. Hence, z cannot equal 11ν unless $\nu=0$.

$l=9$: In this case, $\langle -w_{0,J}(w\cdot 0),\delta^{\vee}\rangle$ and $\langle \lambda,\delta^{\vee}\rangle$ differ by two. There are five ways we can express x in the form $\lambda-a+b+z$. For $\alpha\in\Pi\backslash J$, the bounds on $\langle x,\alpha^{\vee}\rangle$ allow for the possibility that $\langle \nu,\alpha^{\vee}\rangle\neq 0$. Using MAGMA, we compute all possibilities and see that in fact the only solution is when $\nu=0$.

$l=7$: In this case, $\langle -w_{0,J}(w\cdot 0),\delta^{\vee}\rangle$ and $\langle \lambda,\delta^{\vee}\rangle$ differ by one. There are two ways we can express x in the form $\lambda-a+b+z$. For $\alpha\in\Pi\backslash J$, the bounds on $\langle x,\alpha^{\vee}\rangle$ allow for the possibility that $\langle \nu,\alpha^{\vee}\rangle\neq 0$. Using MAGMA, we compute all possibilities and see that in fact the only solution is when $\nu=0$.

$l = 5$: Here $\lambda = -w_{0,J}(w \cdot 0)$. Hence $x = \lambda + z$, and z would need to equal $l\nu$. Here $\Pi \backslash J = \{\alpha_2\}$, and one finds that $0 \leq \langle z, \alpha_2^\vee \rangle \leq 1$. Hence, z cannot equal 5ν unless $\nu = 0$.

Type G_2:

$l = 5$: Here $\lambda = -w_{0,J}(w \cdot 0)$. Hence $x = \lambda + z$, and z would need to equal $l\nu$. However, $\Pi \backslash J = \{\alpha_2\}$, and the only choice for z (other than empty) is $z = (0, 1)$ with $\langle z, \alpha_2^\vee \rangle = 1$. Hence, z cannot equal 5ν unless $\nu = 0$.

5. Cohomology of $\mathrm{H}^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$

The identification of $\mathrm{H}^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$ for small l will proceed in steps. Our computations are motivated by the determination of the support variety $\mathcal{V}_{\mathfrak{g}}(F)$ which may be identified with the restricted nullcone $\mathcal{N}_1(\mathfrak{g})$. By [CLNP], the restricted nullcone $\mathcal{N}_1(\mathfrak{g})$ may be identfied with $G \cdot \mathfrak{u}_J$ for some subset $J \subset \Pi$. When $p \geq h$, we simply have $J = \varnothing$ and $\mathfrak{u}_J = \mathfrak{u}$. To understand the cohomology of $u_\zeta(\mathfrak{g})$, we consider the parabolic subgroup P_J associated to this subset $J \subset \Pi$ with $w(\Phi_0) = \Phi_{w \cdot 0} = \Phi_J$. In Section 5.1, the cohomology of $u_\zeta(\mathfrak{g})$ will be related to that of $u_\zeta(\mathfrak{p}_J)$, and in Section 5.2, the latter cohomology will be related to the cohomology of $u_\zeta(\mathfrak{u}_J)$. Section 5.3 presents a key computation for $\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}_J), \mathbb{C})$ which will then be used in Section 5.5 to compute $\mathrm{H}^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$

Throughout this section, fix $J \subseteq \Pi$. While the goal of this paper is to make cohomological computations in the case that $l < h$, we note that the arguments are also valid for $l \geq h$. In that case, we would have $\Phi_0 = \varnothing$, $J = \varnothing$, and $w = \mathrm{Id}$. Then $P_J = B$, $U_J = U$, $L_J = T$, $\mathfrak{p}_J = \mathfrak{b}$, $\mathfrak{u}_J = \mathfrak{u}$, $\mathfrak{l}_J = \mathfrak{t}$, and the module $M = (\mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^* = \mathbb{C}$. As such, our results encompass the known results for large l in [GK].

5.1. Let $w \in W$ with $w(\Phi_0) = \Phi_J$. By Lemma 4.1.1, $w \cdot 0$ is dominant on J. There exists a spectral sequence (cf. [Jan1, I 4.5]):

$$E_2^{i,j} = R^i \mathrm{ind}_{U_\zeta(\mathfrak{p}_J)}^{U_\zeta(\mathfrak{g})} R^j \mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0 \Rightarrow R^{i+j} \mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{g})} w \cdot 0. \tag{5.1.1}$$

But, $w \cdot 0$ is dominant on J so $R^j \mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0 = 0$ for $j > 0$. This spectral sequence collapses and yields by [A, Cor. 3.8]

$$R^i \mathrm{ind}_{U_\zeta(\mathfrak{p}_J)}^{U_\zeta(\mathfrak{g})} (\mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0) = R^i \mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{g})} w \cdot 0 = \begin{cases} \mathbb{C} & \text{if } i = \ell(w) \\ 0 & \text{if } i \neq \ell(w). \end{cases} \tag{5.1.2}$$

The following spectral sequence provides a connection between the cohomology of $u_\zeta(\mathfrak{p}_J)$ and that of $u_\zeta(\mathfrak{g})$.

Theorem 5.1.1. *Let $w \in W$ such that $w(\Phi_0) = \Phi_{w \cdot 0} = \Phi_J$ where $J \subseteq \Pi$. There exists a first quadrant spectral sequence*

$$E_2^{i,j} = R^i \,\mathrm{ind}_{P_J}^G \mathrm{H}^j(u_\zeta(\mathfrak{p}_J), \mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0) \Rightarrow \mathrm{H}^{i+j-\ell(w)}(u_\zeta(\mathfrak{g}), \mathbb{C}).$$

Proof. We follow the construction in [Jan1, I 6.12]. Consider the functors

$$\mathcal{F}_1(-) = \mathrm{Hom}_{u_\zeta(\mathfrak{g})}(\mathbb{C}, \mathrm{ind}_{U_\zeta(\mathfrak{p}_J)}^{U_\zeta(\mathfrak{g})}(-)) \text{ and } \mathcal{F}_2(-) = \mathrm{ind}_{P_J}^G \mathrm{Hom}_{u_\zeta(\mathfrak{p}_J)}(\mathbb{C}, -)$$

with $\mathcal{F}_1, \mathcal{F}_2 : U_\zeta(\mathfrak{p}_J)$-mod $\to U_\zeta(\mathfrak{g})$-mod (or G-mod). The reader should observe that we are implicitly using the Frobenius map (cf. Section 2.3) and the following identification of functors:

$$\operatorname{ind}_{P_J}^{G}(-) \cong \operatorname{ind}_{U_\zeta(\mathfrak{p}_J)//u_\zeta(\mathfrak{p}_J)}^{U_\zeta(\mathfrak{g})//u_\zeta(\mathfrak{g})}(-).$$

The functors $\mathcal{F}_1$ and $\mathcal{F}_2$ are naturally isomorphic and there exist two spectral sequences:

$$E_2^{i,j} = \mathrm{H}^i(u_\zeta(\mathfrak{g}), R^j\operatorname{ind}_{U_\zeta(\mathfrak{p}_J)}^{U_\zeta(\mathfrak{g})}(\operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0)) \Rightarrow (R^{i+j}\mathcal{F}_1)(\operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0);$$

$$E_2^{i,j} = R^i\operatorname{ind}_{P_J}^{G}\mathrm{H}^j(u_\zeta(\mathfrak{p}_J), \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0) \Rightarrow (R^{i+j}\mathcal{F}_2)(\operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0) \tag{5.1.3}$$

necessarily converging to the same abutment.

By (5.1.2), the first spectral sequence collapses and we can identify

$$(R^{\bullet+\ell(w)}\mathcal{F}_1)(\operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0) \cong \mathrm{H}^\bullet(u_\zeta(\mathfrak{g}), R^{\ell(w)}\operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{g})} w\cdot 0) \cong \mathrm{H}^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C}).$$

Combining this with the second spectral sequence (5.1.3) proves the theorem. □

5.2. We can reidentify the E_2-terms of the spectral sequence in Theorem 5.1.1 by using the spectral sequence in Lemma 2.8.1 for $u_\zeta(\mathfrak{u}_J) \trianglelefteq u_\zeta(\mathfrak{p}_J)$. Note that $u_\zeta(\mathfrak{p}_J)//u_\zeta(\mathfrak{u}_J) \cong u_\zeta(\mathfrak{l}_J)$. We have

$$E_2^{i,j} = \mathrm{H}^i(u_\zeta(\mathfrak{l}_J), \mathrm{H}^j(u_\zeta(\mathfrak{u}_J), \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0)) \Rightarrow \mathrm{H}^{i+j}(u_\zeta(\mathfrak{p}_J), \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0).$$

All of the cohomology groups involved admit an action of $U_\zeta(\mathfrak{p}_J)$ induced from the $\overline{\mathrm{Ad}}$-action. Further, the action of the subalgebra $u_\zeta(\mathfrak{p}_J)$ is trivial. Hence, there is an action of $\mathbb{U}(\mathfrak{p}_J)$ (or equivalently P_J). Moreover, the spectral sequence preserves this action.

Since $\operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0$ is trivial as a $U_\zeta(\mathfrak{u}_J)$-module, the left-hand side of the spectral sequence may be reidentified as follows:

$$E_2^{i,j} = \mathrm{H}^i(u_\zeta(\mathfrak{l}_J), \mathrm{H}^j(u_\zeta(\mathfrak{u}_J), \mathbb{C}) \otimes \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0) \Rightarrow \mathrm{H}^{i+j}(u_\zeta(\mathfrak{p}_J), \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0). \tag{5.2.1}$$

Proposition 5.2.1. *Let $w \in W$ and $J \subset \Pi$ be such that $w(\Phi_0) = \Phi_J$. Then for all $j \geq 0$ there is an isomorphism of $\mathbb{U}(\mathfrak{p}_J)$-modules*

$$\mathrm{H}^j(u_\zeta(\mathfrak{p}_J), \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0) \cong \operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}((\operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0)^*, \mathrm{H}^j(u_\zeta(\mathfrak{u}_J), \mathbb{C})).$$

Proof. Since $\operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0$ is projective as a $u_\zeta(\mathfrak{l}_J)$-module, in the spectral sequence (5.2.1), $E_2^{i,j} = 0$ for all $i0$. Thus the spectral sequence collapses giving for all $j \geq 0$

$$\begin{aligned}\mathrm{H}^j(u_\zeta(\mathfrak{p}_J), \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0) &\simeq \operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}(\mathbb{C}, \mathrm{H}^j(u_\zeta(\mathfrak{u}_J), \mathbb{C}) \otimes \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0)\\ &\simeq \operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}((\operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0)^*, \mathrm{H}^j(u_\zeta(\mathfrak{u}_J), \mathbb{C})).\end{aligned}$$

□

5.3. The following theorem now gives an identification as U_ζ^0-modules (or equivalently $\mathbb{U}(\mathfrak{h})$-modules where $\mathfrak{h} \subset \mathfrak{g}$ is the Cartan subalgebra) of the Hom-group appearing in Proposition 5.2.1.

Theorem 5.3.1. *Let l be as in Assumption 1.3.1 and $w \in W$ such that $w(\Phi_0) = \Phi_J$.*

(a) *Suppose that $l \nmid n+1$ when Φ is of type A_n and $l \neq 9$ when Φ is of type E_6. Then as U_ζ^0-modules*

$$\operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}((\operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0)^*, \operatorname{H}^s(u_\zeta(\mathfrak{u}_J),\mathbb{C})) \cong S^{\frac{s-\ell(w)}{2}}(\mathfrak{u}_J^*)^{[1]}.$$

(b) *If Φ is of type A_n with $n+1 = l(m+1)$ and $w \in W$ is as defined in (4.8.1), then as U_ζ^0-modules*

$$\operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}((\operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0)^*, \operatorname{H}^s(u_\zeta(\mathfrak{u}_J),\mathbb{C})) \cong \bigoplus_{t=0}^{l-1} S^{\frac{s-\ell(w)-(m+1)t(l-t)}{2}}(\mathfrak{u}_J^*)^{[1]} \otimes l\varpi_{t(m+1)},$$

where $\varpi_0 = 0$.

(c) *If Φ is of type E_6 and $l = 9$ (assuming that w and J are as in Section 9.1), then as U_ζ^0-modules*

$$\begin{aligned}\operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}((\operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0)^*, \operatorname{H}^s(u_\zeta(\mathfrak{u}_J),\mathbb{C})) \quad &\cong \quad S^{\frac{s-\ell(w)}{2}}(\mathfrak{u}_J^*)^{[1]} \oplus \left(S^{\frac{s-20}{2}}(\mathfrak{u}_J^*)^{[1]} \otimes l\varpi_1\right) \\ &\oplus \quad \left(S^{\frac{s-20}{2}}(\mathfrak{u}_J^*)^{[1]} \otimes l\varpi_6\right).\end{aligned}$$

Proof. For convenience set $M = (\operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0)^*$ and $\mathcal{G}(-) = \operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}(M,-)$. Since the module M is injective (equivalently projective) as a $u_\zeta(\mathfrak{l}_J)$-module, the functor $\mathcal{G}(-) = \operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}(M,-)$ is an exact functor.

The argument will proceed by induction on successive quotients of $\mathcal{U}_\zeta(\mathfrak{u}_J)$ (cf. [GK, 2.4]). Let $N = |\Phi^+\backslash\Phi_J^+|$. Note that N previously was used to denote $|\Phi^+|$, but this should not cause any confusion here. As in Section 2, choose any fixed ordering of root vectors f_1, f_2, $\dots$, f_N in $\mathcal{U}_\zeta(\mathfrak{u}_J)$ corresponding to the positive roots in $\Phi^+\backslash\Phi_J^+$. For purposes of this argument, the precise ordering is irrelevent. Recall that each f_i^l is central in $\mathcal{U}_\zeta(\mathfrak{u}_J)$. For $0 \leq i \leq N$, set $A_i = \mathcal{U}_\zeta(\mathfrak{u}_J)//\langle f_1^l, f_2^l, \dots, f_i^l\rangle$ where $\langle \dots \rangle$ denotes "the subalgebra generated by $\dots$" (with $A_0 = \mathcal{U}_\zeta(\mathfrak{u}_J)$). Note that $A_N = u_\zeta(\mathfrak{u}_J)$. For $1 \leq i \leq N$, set $B_i = \langle f_i^l\rangle \subset A_{i-1}$ be the subalgebra generated by f_i^l. Note that each B_i is a polynomial algebra in one variable. Note also that B_i is normal (in fact central) in A_{i-1}, and $A_{i-1}//B_i \cong A_i$. For $0 \leq i \leq N$, let V_i be an i-dimensional vector space with basis $\{x_1, x_2, \dots, x_i\}$. Further consider V_i as a U_ζ^0-module by letting x_i have weight γ_i. That is, each x_i is "dual" to the element f_i. In particular, $V_N \cong \mathfrak{u}_J^*$. Here $V_0 = \{0\}$.

Consider first part (a). We prove inductively for $0 \leq i \leq N$ that as U_ζ^0-modules we have

$$\mathcal{G}\big(\operatorname{H}^s(A_i,\mathbb{C})\big) \cong \begin{cases} S^r(V_i)^{[1]} & \text{if } s = 2r + \ell(w) \\ 0 & \text{else.}\end{cases}$$

The case $i = N$ is precisely the statement of the theorem. For $i = 0$, this is precisely Theorem 4.3.1 where by convention we take $S^0(V_0) = \mathbb{C}$.

Assume now that the claim is true for $i-1$, and we will show that it is true for i. We will make use of the LHS spectral sequence of Lemma 2.8.1 for $B_i \trianglelefteq A_{i-1}$. Since B_i is a polynomial algebra in one variable, its cohomology is an exterior algebra in one variable. Precisely, for each i, we have as a U_ζ^0-module:

$$\mathrm{H}^b(B_i,\mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } b=0 \\ \mathbb{C}_i^{[1]} & \text{if } b=1 \\ 0 & \text{else,} \end{cases}$$

where $\mathbb{C}_i$ is a one-dimensional vector space with basis element x_i (i. e., of weight γ_i).

The spectral sequence is

$$E_2^{a,b} = \mathrm{H}^a(A_i, \mathrm{H}^b(B_i,\mathbb{C})) \Rightarrow \mathrm{H}^{a+b}(A_{i-1},\mathbb{C}).$$

The algebra A_i acts on B_i via the $\overline{\mathrm{Ad}}$-action. By Corollary 2.7.4(b), this action is trivial. Therefore this spectral sequence can be rewritten as

$$E_2^{a,b} = \mathrm{H}^a(A_i,\mathbb{C}) \otimes \mathrm{H}^b(B_i,\mathbb{C}) \Rightarrow \mathrm{H}^{a+b}(A_{i-1},\mathbb{C}).$$

Alternately, this easily follows from the above description of $\mathrm{H}^b(B_i,\mathbb{C})$.

Again using the $\overline{\mathrm{Ad}}$-action, $u_\zeta(\mathfrak{l}_J)$ acts on all terms of the spectral sequence and this action is preserved by the differentials. Since the functor $\mathcal{G}(-)$ is exact, we can apply it to the above spectral sequence and obtain a new spectral sequence. For convenience, we abusively use the same name:

$$E_2^{a,b} = \mathcal{G}\big(\mathrm{H}^a(A_i,\mathbb{C}) \otimes \mathrm{H}^b(B_i,\mathbb{C})\big) \Rightarrow \mathcal{G}\big(\mathrm{H}^{a+b}(A_{i-1},\mathbb{C})\big).$$

The above description of $\mathrm{H}^b(B_i,\mathbb{C})$ shows that $u_\zeta(\mathfrak{l}_J)$ acts trivially on it. And hence, this spectral sequence may be rewritten as

$$E_2^{a,b} = \mathcal{G}\big(\mathrm{H}^a(A_i,\mathbb{C})\big) \otimes \mathrm{H}^b(B_i,\mathbb{C}) \Rightarrow \mathcal{G}\big(\mathrm{H}^{a+b}(A_{i-1},\mathbb{C})\big).$$

Observe that $E_2^{a,b} = 0$ for $b \geq 2$. That is, the spectral sequence consists of at most two nonzero rows. So only the first differential $d_2 : E_2^{a,1} \to E_2^{a+2,0}$ could potentially be nonzero. The first row $E_2^{a,0} = \mathrm{H}^a(A_i,\mathbb{C})$ is precisely what we are trying to identify inductively. Note also that for all a, the second row $E_2^{a,1} \cong E_2^{a,0} \otimes \mathbb{C}_i^{[1]}$. In particular, $E_2^{a,1} \neq 0$ if and only if $E_2^{a,0} \neq 0$.

By the inductive hypothesis, we know that the abutment

$$\mathcal{G}\big(\mathrm{H}^{a+b}(A_{i-1},\mathbb{C})\big) \cong \begin{cases} S^r(V_{i-1})^{[1]} & \text{if } a+b = 2r+\ell(w) \\ 0 & \text{else.} \end{cases}$$

In particular, $\mathcal{G}\big(\mathrm{H}^{a+b}(A_{i-1},\mathbb{C})\big) = 0$ for $a+b < \ell(w)$.

Let $A \geq 0$ be least value of a such that $E_2^{a,0} \neq 0$. Hence, A is necessarily the least value of a such that $E_2^{a,1} \neq 0$. In particular, $E_2^{A-2,1} = 0$ and hence $E_\infty^{A,0} \cong E_2^{A,0}/d_2(E_2^{A-2,1}) = E_2^{A,0}$. By the inductive hypothesis, we conclude that $A = \ell(w)$. So for all $a < \ell(w)$ we have

$$\mathcal{G}\big(\mathrm{H}^a(A_i,\mathbb{C})\big) = 0.$$

Next we claim that $E_2^{\ell(w)+a,0} = 0 = E_2^{\ell(w)+a,1}$ for all odd $a > 0$. This can be seen inductively on a. For example, since $E_2^{\ell(w)-1,1} = 0$,

$$E_2^{\ell(w)+1,0} = E_2^{\ell(w)+1,0}/d_2(E_2^{\ell(w)-1,1}) \cong E_\infty^{\ell(w)+1,0} \subset \mathcal{G}\big(\operatorname{H}^{\ell(w)+1}(A_{i-1},\mathbb{C})\big) = 0.$$

Inductively, for odd $a > 0$, we similarly have $E_2^{\ell(w)+a-2,1} = 0$ and so

$$E_2^{\ell(w)+a,0} = E_2^{\ell(w)+a,0}/d_2(E_2^{\ell(w)+a-2,1}) \cong E_\infty^{\ell(w)+a,0} \subset \mathcal{G}\big(\operatorname{H}^{\ell(w)+a}(A_{i-1},\mathbb{C})\big) = 0.$$

In other words, for all odd $a > 0$, we have (as claimed)

$$\mathcal{G}\big(\operatorname{H}^{\ell(w)+a}(A_i,\mathbb{C})\big) = 0.$$

Summarizing: our spectral sequence has $E_2^{a,b} = 0$ for $a < \ell(w)$ and $E_2^{\ell(w)+a,b} = 0$ for all odd $a > 0$. That is, the columns are initially zero and then begin to alternate nonzero (potentially) and zero thereafter.

Further, for all even $a \geq 0$, we then have

$$\ker\{d_2 : E_2^{\ell(w)+a,1} \to E_2^{\ell(w)+a+2,0}\} \subset E_\infty^{\ell(w)+a,1} \subset \mathcal{G}\big(\operatorname{H}^{\ell(w)+a+1}(A_{i-1},\mathbb{C})\big) = 0.$$

Therefore $d_2 : E_2^{\ell(w)+a,1} \to E_2^{\ell(w)+a+2,0}$ is injective. Hence, we have for all even $a \geq 0$,

$$\begin{aligned} E_2^{\ell(w)+a,0}/E_2^{\ell(w)+a-2,1} &\cong E_2^{\ell(w)+a,0}/d_2(E_2^{\ell(w)+a-2,1}) \cong E_\infty^{\ell(w)+a,0} \\ &\cong \mathcal{G}\big(\operatorname{H}^{\ell(w)+a}(A_{i-1},\mathbb{C})\big) = S^{a/2}(V_{i-1})^{[1]}. \end{aligned}$$

So we have a short exact sequence of U_ζ^0-modules:

$$0 \to E_2^{\ell(w)+a-2,1} \to E_2^{\ell(w)+a,0} \to S^{a/2}(V_{i-1})^{[1]} \to 0.$$

But identifying these E_2-terms gives

$$0 \to \mathcal{G}\big(\operatorname{H}^{\ell(w)+a-2}(A_i,\mathbb{C})\big) \otimes \mathbb{C}_i^{[1]} \to \mathcal{G}\big(\operatorname{H}^{\ell(w)+a}(A_i,\mathbb{C})\big) \to S^{a/2}(V_{i-1})^{[1]} \to 0.$$

Inducting now on even $a \geq 0$, we may assume that

$$\mathcal{G}\big(\operatorname{H}^{\ell(w)+a-2}(A_i,\mathbb{C})\big) = S^{(a-2)/2}(V_i)^{[1]}.$$

The short exact sequence becomes

$$0 \to S^{(a-2)/2}(V_i)^{[1]} \otimes \mathbb{C}_i^{[1]} \to \mathcal{G}\big(\operatorname{H}^{\ell(w)+a}(A_i,\mathbb{C})\big) \to S^{a/2}(V_{i-1})^{[1]} \to 0$$

or setting $a = 2r$,

$$0 \to S^{r-1}(V_i)^{[1]} \otimes \mathbb{C}_i^{[1]} \to \mathcal{G}\big(\operatorname{H}^{\ell(w)+2r}(A_i,\mathbb{C})\big) \to S^{r}(V_{i-1})^{[1]} \to 0.$$

Hence as a U_ζ^0-module,

$$\mathcal{G}\big(\operatorname{H}^{\ell(w)+2r}(A_i,\mathbb{C})\big) \cong (S^{r-1}(V_i)^{[1]} \otimes \mathbb{C}_i^{[1]}) \oplus S^{r}(V_{i-1})^{[1]}.$$

The left hand factor consists of r-fold symmetric powers in the x_j which contain at least one x_i, while the righthand factor consists of r-fold symmetric powers in x_j with $1 \leq j \leq i-1$. Hence

$$\mathcal{G}\big(\operatorname{H}^{\ell(w)+2r}(A_i,\mathbb{C})\big) \cong S^{r}(V_i)^{[1]}$$

which along with the above conclusions verifies the inductive claim and hence part (a) of the theorem.

For parts (b) and (c) a similar argument can be used whose details are left to the interested reader. Of crucial importance here is the degrees in which the "extra" classes arise in parts (b) and (c) of Theorem 4.3.1. For example, consider part (b). We have $\mathcal{G}\big(\operatorname{H}^i(\mathcal{U}_\zeta(\mathfrak{u}_J),\mathbb{C})\big)=0$ unless $i=\ell(w)+(m+1)t(l-t)$ for $0\le t\le l-1$. Observe that $t(l-t)$ (and, hence, $(m+1)t(l-t)$) is necessarily even. Thus the extra cohomology classes appear in degrees having the same parity as $\ell(w)$. As such, in the above argument, we will have a similar phenomenon happening in the spectral sequence: $E_2^{a,b}=0$ for $a<\ell(w)$ or $a=\ell(w)+a'$ with $a'>0$ being odd, and the analogous argument will give the claim. Similary in part (c), we see that the extra cohomology classes lie in a degree with the same parity as $\ell(w)$.

□

5.4. As mentioned previously, the Hom-groups in Theorem 5.3.1 admit an action of $\mathbb{U}(\mathfrak{p}_J)$ induced from the $\overline{\text{Ad}}$-action. On the other hand $\mathbb{U}(\mathfrak{p}_J)$ acts naturally on $\mathfrak{u}_J$ by the adjoint action or on $\mathfrak{u}_J^*$ by the coadjoint action. This can be further extended to an action on $S^\bullet(\mathfrak{u}_J^*)$. With this action, the isomorphisms in Theorem 5.3.1 also hold as $\mathbb{U}(\mathfrak{p}_J)$-modules (or equivalently P_J-modules).

Lemma 5.4.1. *The isomorphisms of Theorem 5.3.1 also hold as $\mathbb{U}(\mathfrak{p}_J)$-modules where the actions are as described above.*

Proof. Consider first the generic case - part (a). We use notation as in the proof of Theorem 5.3.1. Let $Z_J=\langle f_1^l,\dots,f_N^l\rangle\subset\mathcal{U}_\zeta(\mathfrak{u}_J)$ be the central subalgebra such that $\mathcal{U}_\zeta(\mathfrak{u}_J)//Z_J\cong u_\zeta(\mathfrak{u}_J)$ (cf. also Section 2.7). Consider the spectral sequence

$$E_2^{a,b}=\operatorname{H}^a(u_\zeta(\mathfrak{u}_J),\operatorname{H}^b(Z_J,\mathbb{C}))\Rightarrow\operatorname{H}^{a+b}(\mathcal{U}_\zeta(\mathfrak{u}_J),\mathbb{C})$$

given by Lemma 2.8.1. This is a spectral sequence of $U_\zeta(\mathfrak{p}_J)$-modules. Since $u_\zeta(\mathfrak{p}_J)$ acts trivially on $\operatorname{H}^b(Z_J,\mathbb{C})$ (cf. Corollary 2.7.4(b)), this may be rewritten as

$$E_2^{a,b}=\operatorname{H}^a(u_\zeta(\mathfrak{u}_J),\mathbb{C})\otimes\operatorname{H}^b(Z_J,\mathbb{C}))\Rightarrow\operatorname{H}^{a+b}(\mathcal{U}_\zeta(\mathfrak{u}_J),\mathbb{C}).$$

Further, applying the functor $\mathcal{G}(-)$ we get a new spectral sequence (using the same name)

$$E_2^{a,b}=\mathcal{G}\big(\operatorname{H}^a(u_\zeta(\mathfrak{u}_J),\mathbb{C})\big)\otimes\operatorname{H}^b(Z_J,\mathbb{C})\Rightarrow\mathcal{G}\big(\operatorname{H}^{a+b}(\mathcal{U}_\zeta(\mathfrak{u}_J),\mathbb{C})\big),$$

whose differentials still preserve the action of $U_\zeta(\mathfrak{p}_J)$. Since $u_\zeta(\mathfrak{u}_J)$ acts trivially on both $\operatorname{H}^\bullet(u_\zeta(\mathfrak{u}_J),\mathbb{C})$ and $\operatorname{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J),\mathbb{C})$, and $u_\zeta(\mathfrak{l}_J)\cong u_\zeta(\mathfrak{p}_J)//u_\zeta(\mathfrak{u}_J)$, we have

$$\mathcal{G}\big(\operatorname{H}^\bullet(u_\zeta(\mathfrak{u}_J),\mathbb{C})\big)=\operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}(M,\operatorname{H}^\bullet(u_\zeta(\mathfrak{u}_J),\mathbb{C}))\cong\operatorname{Hom}_{u_\zeta(\mathfrak{p}_J)}(M,\operatorname{H}^\bullet(u_\zeta(\mathfrak{u}_J),\mathbb{C}))$$

and

$$\mathcal{G}\big(\operatorname{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J),\mathbb{C})\big)=\operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}(M,\operatorname{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J),\mathbb{C}))\cong\operatorname{Hom}_{u_\zeta(\mathfrak{p}_J)}(M,\operatorname{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J),\mathbb{C})).$$

Therefore, $u_\zeta(\mathfrak{p}_J)$ acts trivially on this new spectral sequence, and so this is a spectral sequence of $\mathbb{U}(\mathfrak{p}_J)$-modules.

By Theorem 4.3.1, the abutment is nonzero only when $a+b=\ell(w)$ in which case it is the trivial module $\mathbb{C}$. And by Theorem 5.3.1, as $\mathbb{U}(\mathfrak{h})$-modules, we know that $\mathcal{G}\big(\operatorname{H}^a(u_\zeta(\mathfrak{u}_J),\mathbb{C})\big)\cong S^{\frac{a-\ell(w)}{2}}(\mathfrak{u}_J^*)$. Also, the algebra Z_J is a polynomial algebra and so its cohomology as an algebra is an exterior algebra. Moreover, by [ABG, Cor. 2.9.6], as $\mathbb{U}(\mathfrak{b})$-modules, $\operatorname{H}^\bullet(Z_J,\mathbb{C})\cong\Lambda^\bullet(\mathfrak{u}_J^*)$ (i. e., the ordinary exterior algebra on $\mathfrak{u}_J^*$), where the action of $\mathbb{U}(\mathfrak{b})$ on $\Lambda^\bullet(\mathfrak{u}_J^*)$ is given by the coadjoint action. Since $\mathrm{H}^\bullet(Z_J,\mathbb{C})$ is a $\mathbb{U}(\mathfrak{p}_J)$-module and the coadjoint action on $\Lambda^\bullet(\mathfrak{u}_J^*)$ can be extended to $\mathfrak{p}_J$, by applying $\operatorname{ind}_{\mathbb{U}(\mathfrak{b})}^{\mathbb{U}(\mathfrak{p}_J)}(-)$, we see that $\mathrm{H}^\bullet(Z_J,\mathbb{C})\cong\Lambda^\bullet(\mathfrak{u}_J^*)$ as $\mathfrak{p}_J$-modules.

As in Theorem 5.3.1, $E_2^{a,b}=0$ for $a<\ell(w)$ and $E_2^{\ell(w)+a,b}=0$ for odd a. Consider the term

$$E_2^{\ell(w)+2,0}=\mathcal{G}\big(\,\mathrm{H}^{\ell(w)+2}(u_\zeta(\mathfrak{u}_J),\mathbb{C})\big)\cong S^1(\mathfrak{u}_J^*)=\mathfrak{u}_J^*,$$

where the latter identifications are as $\mathbb{U}(\mathfrak{h})$-modules. We will see this also holds as $\mathbb{U}(\mathfrak{p}_J)$-modules. Since $E_\infty^{\ell(w),1}=0$ and $E_\infty^{\ell(w)+2,0}=0$, the differential $d_2: E_2^{\ell(w),1}\to E_2^{\ell(w)+2,0}$ is an isomorphism of $\mathbb{U}(\mathfrak{p}_J)$-modules.

Since $E_2^{\ell(w),1}=\mathrm{H}^1(Z_J,\mathbb{C})\cong\mathfrak{u}_J^*$ as a $\mathbb{U}(\mathfrak{p}_J)$-module and $E_2^{\ell(w)+2,0}=\mathcal{G}\big(\,\mathrm{H}^{\ell(w)+2}(u_\zeta(\mathfrak{u}_J),\mathbb{C})\big)\cong\mathfrak{u}_J^*$ as a $\mathbb{U}(\mathfrak{h})$-module, this latter identification must also hold as a $\mathbb{U}(\mathfrak{p}_J)$-module.

As already noted, the $\mathbb{U}(\mathfrak{p}_J)$-structure of the terms

$$E_2^{\ell(w),b}=\mathrm{H}^b(Z_J,\mathbb{C})\cong\Lambda^b(\mathfrak{u}_J^*)$$

is determined by the coadjoint action. Further, we can now say that the $\mathbb{U}(\mathfrak{p}_J)$-structure of the terms

$$E_2^{\ell(w)+2,b}=\mathcal{G}\big(\,\mathrm{H}^{\ell(w)+2}(u_\zeta(\mathfrak{u}_J),\mathbb{C})\big)\otimes\mathrm{H}^b(Z_J,\mathbb{C})\cong\mathfrak{u}_J^*\otimes\Lambda^b(\mathfrak{u}_J^*)$$

is determined by the coadjoint action. Therefore, since $E_\infty^{\ell(w)+4,0}=0$, the $\mathbb{U}(\mathfrak{p}_J)$-structure of

$$E_2^{\ell(w)+4,0}=\mathcal{G}\big(\,\mathrm{H}^{\ell(w)+4}(u_\zeta(\mathfrak{u}_J),\mathbb{C})\big)\cong S^2(\mathfrak{u}_J^*)$$

(with isomorphism as a $\mathbb{U}(\mathfrak{h})$-module), which is determined by the structure on $E_2^{\ell(w),b}$ and $E_2^{\ell(w)+2,b}$, must also be determined by the coadjoint action. Inductively, we conclude that indeed the $\mathbb{U}(\mathfrak{p}_J)$-action on

$$\mathcal{G}\big(\,\mathrm{H}^a(u_\zeta(\mathfrak{u}_J),\mathbb{C})\big)\cong S^{\frac{a-\ell(w)}{2}}(\mathfrak{u}_J^*)$$

is given by the coadjoint action.

For parts (b) and (c), a similar argument may be used. The key here (as in Theorem 5.3.1) is that the additional classes in $\mathcal{G}\big(\,\mathrm{H}^i(\mathfrak{u}_J,\mathbb{C})\big)$ lie in degrees which have the same parity as $\ell(w)$. Further, the additional classes have distinct (nonzero) weights which are not strongly linked as noted in Remark 4.3.2. □

5.5. Proof of Theorem 1.3.2. Let us recall the following vanishing result due to Broer [Br2, Thm. 2.2] and extended to a more general setting by Sommers [So1, Prop. 4.2]. Let J be an arbitrary subset of Π. Then

$$R^i\operatorname{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*)\otimes\lambda=0 \tag{5.5.1}$$

for $i>0$ and $\lambda\in X(P_J)_+$ where $X(P_J)_+$ is the set of dominant weights inside the character group $X(P_J)$. This vanishing result is proved using the Grauert-Riemenschneider theorem. We can now prove the first of our main theorems which provides a precise description of the cohomology of quantum groups at roots of unity in the case that ζ is a primitive lth root of unity.

Proof. (i) According to Proposition 5.2.1, Theorem 5.3.1(a), and Lemma 5.4.1, we have, as $\mathbb{U}(\mathfrak{p}_J)$-modules,

$$\mathrm{H}^j(u_\zeta(\mathfrak{p}_J), \mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w\cdot 0) \cong S^{\frac{j-\ell(w)}{2}}(\mathfrak{u}_J^*).$$

By substituting this identification into the spectral sequence given in Theorem 5.1.1, we have

$$E_2^{i,j} = R^i \operatorname{ind}_{P_J}^G S^{\frac{j-\ell(w)}{2}}(\mathfrak{u}_J^*) \Rightarrow \mathrm{H}^{i+j-\ell(w)}(u_\zeta(\mathfrak{g}), \mathbb{C}).$$

We can now apply (5.5.1) to conclude that $E_2^{i,j} = 0$ for $i > 0$, thus the spectral sequence collapses to yield

$$\mathrm{H}^s(u_\zeta(\mathfrak{g}), \mathbb{C}) = \operatorname{ind}_{P_J}^G S^{\frac{s}{2}}(\mathfrak{u}_J^*).$$

This gives part (a) for those types listed in (b)(i). According to Theorem 3.7.1, $\operatorname{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*) \cong \mathbb{C}[G\cdot\mathfrak{u}_J] \cong \mathbb{C}[\mathcal{N}(\Phi_0)]$, which gives part (b)(i).

(ii) In this case we have the following spectral sequence (using Proposition 5.2.1, Theorem 5.3.1(b), and Lemma 5.4.1),

$$E_2^{i,j} = \bigoplus_{t=0}^{l-1} R^i \operatorname{ind}_{P_J}^G S^{\frac{j-\ell(w)-(m+1)t(l-t)}{2}}(\mathfrak{u}_J^*) \otimes \varpi_{t(m-1)} \Rightarrow \mathrm{H}^{i+j-\ell(w)}(u_\zeta(\mathfrak{g}), \mathbb{C}).$$

One can again apply (5.5.1) because we are tensoring the symmetric algebra by weights in $X(P_J)_+$, thus the spectral sequence collapses and yields part (ii) of (a) and (b). One can argue similarly for part (iii) using Theorem 5.3.1(c). In that case $\ell(w) = 8$. □

6. Finite Generation.

This section provides provide a proof of Theorem 1.3.4 in Section 1.3.

6.1. For part (a), if $l \nmid n+1$ when Φ is of type A_n, $l \neq 9$ when Φ is of type E_6, and $l \neq 7, 9$ when Φ is of type E_8, then $\mathrm{H}^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$ is by Theorem 1.3.2 the coordinate algebra of the affine variety $\mathcal{N}(\Phi_0)$, so the cohomology algebra is a finitely generated $\mathbb{C}$-algebra. If $l \mid n+1$ when Φ is of type A_n or $l = 9$ when Φ is of type E_6, we have $\mathrm{H}^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C}) \cong \bigoplus \operatorname{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*) \otimes B_t$ being a finite sum where each B_t is finite dimensional. Thus, the algebra $R := \mathrm{H}^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$ is finitely generated as a module over the subring $\operatorname{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*) \cong \mathbb{C}[\mathcal{N}(\Phi_0)]$ so, by the Hilbert Basis Theorem, R is finitely generated. Lastly, if $l = 7, 9$ when Φ is of type E_8, then $\mathrm{H}^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C}) \cong \operatorname{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*) \cong \mathbb{C}[G \times_{P_J} \mathfrak{u}_J]$ (cf. Section 3.7). Let $\mathcal{O}$ denote the corresponding nilpotent orbit (cf. Section 9.1). That is, $G\cdot\mathfrak{u}_J = \overline{\mathcal{O}}$. Since the moment map $G \times_{P_J} \mathfrak{u}_J \to G\cdot\mathfrak{u}_J$ is a resolution of singularities (that portion of the proof of Theorem 3.7.1 still holds), $\mathbb{C}[G \times_{P_J} \mathfrak{u}_J] \cong \mathbb{C}[\mathcal{O}]$ (cf. [KP] or [Sol]). We have $\mathbb{C}[\overline{\mathcal{O}}] \subseteq \mathbb{C}[\mathcal{O}]$ and, by [Jan3, Prop. 8.3], $\mathbb{C}[\mathcal{O}]$ is the integral closure of $\mathbb{C}[\overline{\mathcal{O}}]$ in its field of fractions. Since $\mathbb{C}[\overline{\mathcal{O}}] = \mathbb{C}[G\cdot\mathfrak{u}_J]$ is finitely generated, $\mathbb{C}[\mathcal{O}] \cong \mathbb{C}[G \times_{P_J} \mathfrak{u}_J] \cong \mathrm{H}^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$ is also finitely generated by [Mat, p. 261-63] and [Kun, Cor. 2.4].

Remark 6.1.1. Part (a) of Theorem 1.3.4 holds if l is as in Assumption 1.3.1.

6.2. Part (b) will be proved in Section 6.3. We first make some preliminary observations. In Section 2.9, a filtration was introduced on $\mathcal{U}_\zeta(\mathfrak{u}_J)$ which can be restricted to $u_\zeta(\mathfrak{u}_J)$. Since $u_\zeta(\mathfrak{u}_J)$ is finite dimensional, the induced filtration on the cobar complex computing the cohomology $\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}_J),\mathbb{C})$ is finite. As in the proof of part (b) of Proposition 2.9.1, there is a spectral sequence as follows.

Lemma 6.2.1. *There is a spectral sequence*

$$E_1^{i,j} = \mathrm{H}^{i+j}(\mathrm{gr}\, u_\zeta(\mathfrak{u}_J),\mathbb{C})_{(i)} \Rightarrow \mathrm{H}^{i+j}(u_\zeta(\mathfrak{u}_J),\mathbb{C})$$

of graded algebras and U_ζ^0-modules.

Note that since the filtration on $u_\zeta(\mathfrak{u}_J)$ is finite, this spectral sequence has only finitely many columns, and hence eventually stops.

Globally, we have $E_1^{\bullet,\bullet} \cong \mathrm{H}^\bullet(\mathrm{gr}\, u_\zeta(\mathfrak{u}_J),\mathbb{C})$. By [GK, Prop. 2.3.1], there exists a graded algebra isomorphism

$$\mathrm{H}^\bullet(\mathrm{gr}\, u_\zeta(\mathfrak{u}_J),\mathbb{C}) \cong S^\bullet(\mathfrak{u}_J^*) \otimes \Lambda_{\zeta,J}^\bullet. \tag{6.2.1}$$

This is also an isomorphism of U_ζ^0-modules with u_ζ^0 acting trivially on the symmetric algebra. Further, under the isomorphism (6.2.1),

$$\mathrm{H}^n(\mathrm{gr}\, u_\zeta(\mathfrak{u}_J),\mathbb{C}) \cong \bigoplus_{2a+b=n} S^a(\mathfrak{u}_J^*)^{[1]} \otimes \Lambda_{\zeta,J}^b. \tag{6.2.2}$$

By the isomorphism (6.2.1), $E_1^{\bullet,\bullet}$ is finitely generated over a subalgebra which is isomorphic (as algebras and U_ζ^0-modules) to $S^\bullet(\mathfrak{u}_J^*)^{[1]}$. For notational convenience, we abusively consider $S^\bullet(\mathfrak{u}_J^*)^{[1]}$ as a subalgebra of $E_1^{\bullet,\bullet}$. Under mild conditions on l, this subalgebra consists of universal cycles.

Proposition 6.2.2. *Let l satisfy Assumption 1.3.3 and $J \subseteq \Pi$. In the spectral sequence of Lemma 6.2.1, $d_1(S^\bullet(\mathfrak{u}_J^*)^{[1]}) = 0$.*

Proof. From (6.2.2), the submodule $S^1(\mathfrak{u}_J^*)^{[1]}$ is identified with a submodule of $\mathrm{H}^2(\mathrm{gr}\, u_\zeta(\mathfrak{u}_J),\mathbb{C})$. As such, the image of $S^1(\mathfrak{u}_J^*)^{[1]}$ under d_1 must lie in

$$\mathrm{H}^3(\mathrm{gr}\, u_\zeta(\mathfrak{u}_J),\mathbb{C}) \cong (S^1(\mathfrak{u}_J^*)^{[1]} \otimes \Lambda_{\zeta,J}^1) \oplus \Lambda_{\zeta,J}^3.$$

A U_ζ^0-homogeneous element x_σ of $S^1(\mathfrak{u}_J^*)^{[1]}$ has weight $l\sigma$ for some $\sigma \in \Phi^+ - \Phi_J^+$. Hence, by weight considerations, if the image of x_σ is not zero, it cannot lie in $S^1(\mathfrak{u}_J^*)^{[1]} \otimes \Lambda_{\zeta,J}^1$. On the other hand, for x_σ to have nonzero image in $\Lambda_{\zeta,k}^3$, we must have $l\sigma = \gamma_1 + \gamma_2 + \gamma_3$ for three distinct (positive) roots $\gamma_i \in \Phi^+ - \Phi_J^+$. Under the given conditions on l, this is not possible. To see this, we argue by the type of Φ.

For any weight η which lies in the positive root lattice, we can write $\eta = \sum_{\beta\in\Pi} n_{\eta,\beta}\beta$ for unique $n_{\eta,\beta} \in \mathbb{Z}_{\geq 0}$. Set $\gamma := \gamma_1 + \gamma_2 + \gamma_3$ for γ_i as above. In order to have $\gamma = l\sigma$, l must divide $n_{\gamma,\beta}$ for each $\beta \in \Pi$.

In type A_n, for each γ_i and $\beta \in \Pi$, we have $n_{\gamma_i,\beta} \leq 1$. Hence $n_{\gamma,\beta} \leq 3$, and the claim immediately follows for $l > 3$. For $l = 3$, we could only have $3\sigma = \gamma$ if $\gamma_1 = \gamma_2 = \gamma_3$ which contradicts our assumption.

For types B_n, C_n, and D_n, $n_{\gamma,\beta} \leq 6$. Hence the claim follows if $l \geq 7$. Suppose $l = 5$ and $n_{\gamma,\beta} = 5$ for some $\beta \in \Pi$. Without a loss of generality we may assume that $n_{\gamma_1,\beta} = 2$, $n_{\gamma_2,\beta} = 2$, and $n_{\gamma_3,\beta} = 1$. Observe that there is necessarily some $\beta' \in \Pi$ such that $n_{\gamma_1,\beta'} = 1$, $n_{\gamma_2,\beta'} = 1$, and $n_{\gamma_3,\beta'} \in \{0,1\}$. Thus $n_{\gamma,\beta'} \in \{2,3\}$ and so cannot be divisible by 5. In types B_n and C_n when $l = 3$, the condition can be satisfied (e.g., in type B_2, $\alpha_1 + (\alpha_1 + \alpha_2) + (\alpha_1 + 2\alpha_2) = 3(\alpha_1 + \alpha_2)$).

On the other hand, in type D_n, the condition is still not possible when $l = 3$. To see this, suppose on the contrary that $3\sigma = \gamma_1 + \gamma_2 + \gamma_3 = \gamma$. If $n_{\gamma_i,\beta} \leq 1$ for each γ_i and all $\beta \in \Pi$, then we are done as in type A_n. Suppose now that $n_{\gamma_1,\beta} = 2$ for some $\beta \in \Pi$. Recall that if η is a positive root with $n_{\eta,\beta} = 2$, then (in the standard Bourbaki ordering)

$$\eta = \alpha_i + \cdots \alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$$

for some $j \geq i$. Hence we have $n_{\gamma_1,\alpha_{n-2}} = 2$, $n_{\gamma_1,\alpha_{n-1}} = 1$ and $n_{\gamma_1,\alpha_n} = 1$. In order to have $n_{\gamma,\alpha_{n-1}} = 3 = n_{\gamma,\alpha_n}$, we must also have $n_{\gamma_2,\alpha_{n-1}} = n_{\gamma_2,\alpha_n} = n_{\gamma_3,\alpha_{n-1}} = n_{\gamma_3,\alpha_n} = 1$. To have $n_{\gamma,\alpha_{n-2}}$ divisible by 3, there two cases to consider: either $n_{\gamma_2,\alpha_{n-2}} = 1$ and $n_{\gamma_3,\alpha_{n-2}} = 0$ or $n_{\gamma_2,\alpha_{n-2}} = 2 = n_{\gamma_3,\alpha_{n-2}}$. However, there are no such roots γ_3 satisifying $n_{\gamma_3,\alpha_{n-2}} = 0$, $n_{\gamma_3,\alpha_{n-1}} = 1$, and $n_{\gamma_3,\alpha_n} = 1$. So the first case is not possible.

Suppose the latter case holds. Then for each i, $n_{\gamma_i,\alpha_{n-3}} \in \{1,2\}$. If these numbers are not all 2 or all 1, then we are done. If each $n_{\gamma_i,\alpha_{n-3}} = 2$, inductively computing $n_{\gamma_i,\beta}$ for $\beta = \alpha_m$ with $m < n-3$ either we are done or we come to the case that $n_{\gamma_i,\beta} = 1$ for each i. Continuing from that case, since the γ_i are distinct, there is some $\beta' = \alpha_{m'}$ with $m' < m$ such that $n_{\gamma_1,\beta'} = 1$, $n_{\gamma_2,\beta'} \in \{0,1\}$, and $n_{\gamma_3,\beta'} = 0$. Hence $n_{\gamma,\beta'}$ is not divisible by 3, and we are done.

For the exceptional types, one can check "by hand", using for example MAGMA, that the root condition $l\sigma = \gamma_1 + \gamma_2 + \gamma_3$ cannot hold for $l > 3$.

Hence, under our assumptions on l, we must have $d_1(S^1(\mathfrak{u}_J^*)^{[1]}) = 0$. Since the differentials in the spectral sequence are derivations with respect to the cup product, it follows that $d_1(S^\bullet(\mathfrak{u}_J^*)^{[1]}) = 0$. □

6.3. We now prove part (b) of Theorem 1.3.4. Let M be a finite dimensional $u_\zeta(\mathfrak{g})$-module. Without a loss of generality, we may assume that M is an irreducible $u_\zeta(\mathfrak{g})$-module because of the following proposition which is easily proved by using induction on the composition length of the module and the long exact sequence in cohomology.

Proposition 6.3.1. *Let* $R := \mathrm{H}^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$ *and* M *be a finite dimensional* $u_\zeta(\mathfrak{g})$*-module. Suppose that* $\mathrm{H}^\bullet(u_\zeta(\mathfrak{g}), S)$ *is finitely generated over* R *for all irreducible* $u_\zeta(\mathfrak{g})$*-modules* S*. Then* $\mathrm{H}^\bullet(u_\zeta(\mathfrak{g}), M)$ *is finitely generated over* R*.*

Let S be an irreducible $u_\zeta(\mathfrak{g})$-module. Since S lifts to a $U_\zeta(\mathfrak{g})$-module, there exists a spectral sequence of $R = \mathrm{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*)$-modules (obtained in a manner analogous to that of Theorem 5.1.1):

$$E_2^{i,j} = R^i \mathrm{ind}_{P_J}^G \mathrm{H}^j(u_\zeta(\mathfrak{p}_J), \mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0 \otimes S) \Rightarrow \mathrm{H}^{i+j-\ell(w)}(u_\zeta(\mathfrak{g}), S).$$

Using this spectral sequence, it suffices to show that $D := \mathrm{H}^\bullet(u_\zeta(\mathfrak{p}_J), \mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0 \otimes S)$ is finitely generated over $S^\bullet(\mathfrak{u}_J^*)^{[1]}$ because then $E_2^{i,\bullet}$ is finitely generated over R for each i. Further, this spectral sequence stops (i.e., $E_r = E_\infty$ for r sufficiently large) because the higher right derived functors $R^i\text{ind}_{P_J}^G-$ vanish when $i > \dim G/P_J$. Thus E_∞ is finitely generated over R, and $\text{H}^\bullet(u_\zeta(\mathfrak{g}), S)$ is finitely generated over R.

To show that D is finitely generated over $S^\bullet(\mathfrak{u}_J^*)^{[1]}$, observe that one can use the LHS spectral sequence (Lemma 2.8.1) to show that

$$D \cong \text{Hom}_{u_\zeta(\mathfrak{l}_J)}((\text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^*, \text{H}^\bullet(u_\zeta(\mathfrak{u}_J), S)).$$

By the same principles as used in the proposition above, it suffices to show finite generation when S is an irreducible $u_\zeta(\mathfrak{p}_J)$-module. Note that irreducible $u_\zeta(\mathfrak{p}_J)$-modules are obtained by inflating irreducible $u_\zeta(\mathfrak{l}_J)$-modules, so $u_\zeta(\mathfrak{u}_J)$ acts trivially on S. Now we have

$$D \cong \text{Hom}_{u_\zeta(\mathfrak{l}_J)}((\text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^* \otimes S^*, \text{H}^\bullet(u_\zeta(\mathfrak{u}_J), \mathbb{C})).$$

But, $(\text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^* \otimes S^*$ is a projective $u_\zeta(\mathfrak{l}_J)$-module. Thus we need to show that $D_P := \text{Hom}_{u_\zeta(\mathfrak{l}_J)}(P, \text{H}^\bullet(u_\zeta(\mathfrak{u}_J), \mathbb{C}))$ is finitely generated over $S^\bullet(\mathfrak{u}_J^*)^{[1]}$ where P is an arbitrary projective indecomposable $u_\zeta(\mathfrak{l}_J)$-module. Observe that $\text{H}^\bullet(u_\zeta(\mathfrak{u}_J), \mathbb{C}) \cong \oplus_P D_P$ where the sum is taken over all projective indecomposable $u_\zeta(\mathfrak{l}_J)$-modules. We are now reduced to proving that $\text{H}^\bullet(u_\zeta(\mathfrak{u}_J), \mathbb{C})$ is a finitely generated $S^\bullet(\mathfrak{u}_J^*)^{[1]}$-module.

Consider the spectral sequence of Lemma 6.2.1 and the first differential d_1. By Proposition 6.2.2, $d_1(S^\bullet(\mathfrak{u}_J^*)^{[1]}) = 0$, so by the argument given in [Ba], it follows that $\text{H}^\bullet(u_\zeta(\mathfrak{u}_J), \mathbb{C})$ is finitely generated over $S^\bullet(\mathfrak{u}_J^*)^{[1]}$.

7. Frobenius kernels

7.1. Let G be a reductive connected algebraic group scheme defined over $\mathbb{F}_p$ and F be an algebraically closed field of characteristic $p > 0$. Let $\text{Fr} : G \to G$ be the Frobenius map and $G_r = \ker \text{Fr}^r$ where Fr^r is the composition of Fr with itself r-times. Friedlander and Parshall [FP2] computed $\text{H}^\bullet(G_1, F)$ for $p \geq 3(h-1)$ by showing that $\text{H}^{2\bullet}(G_1, F)$ can be identified with the coordinate algebra of $\mathcal{N}$ and that the odd degree cohomology vanishes. Andersen and Jantzen [AJ] provided a different proof and showed this also holds when $p > h$. In the same paper, they provide some ad hoc calculations of the G_1-cohomology for small primes. In this section we will demonstrate how these calculations fit into our general set up.

Given a rational G-module M, let $M^{(1)}$ denote the module obtained by first applying the Frobenius morphism to G and then acting as given. Note that the Frobenius kernel G_1 acts trivially on such a module. Conversely, given a rational G-module N on which G_1 acts trivially, there exists a rational G-module M such that $N = M^{(1)}$. We write $M = N^{(-1)}$ (cf. [Jan1, II 3.16]).

7.2. For $J \subset \Pi$, we formulate two assumptions. The first assumption involves Grauert-Riemenschneider vanishing:

(A1) $R^i\text{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J{}^*) = 0$ for $i > 0$.

(A1)′ $R^i\text{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J{}^*) \otimes \lambda = 0$ for $i > 0$ and $\lambda \in X(P_J)_+$.

The second assumption on J is a condition on the normality of the closure of the Richardson orbit defined by J:

(A2) The variety $G \cdot \mathfrak{u}_J$ is normal.

These two assumptions hold in the case when the field is $\mathbb{C}$ and were used to compute the cohomology for quantum groups. When F is an algebraically closed field of characterstic $p > 0$, much less is known about the validity of (A1) and (A2). In this situation, the nullcone $\mathcal{N}$ is still normal and Kumar, Lauritzen, and Thomsen [KLT] have shown that the closure of the subregular orbit is normal. Thomsen [Th, Prop. 7] has also shown that closures of Richardson orbits whose corresponding set J consists of a set of pairwise orthogonal short simple roots are normal. Donkin [D] proved that all orbit closures in type A_n are normal. Christophersen [C] has recently determined the nilpotent orbits for type E_6 which have normal orbit closure when $p \geq 5$.

7.3. One can now use the assumptions (A1) and (A2) to formulate when the cohomology algebra for G_1 identifies with the coordinate algebra on the restricted nullcone. Note that the proof here is simpler than in the quantum group setting due to the fact that the exterior algebra $\Lambda^\bullet(\mathfrak{u}_J^*)$ has a natural structure as a P_J-module whereas the quantized exterior algebra $\Lambda^\bullet_{\zeta,J}$ does not admit a natural structure as a $U_\zeta(\mathfrak{p}_J)$-module.

Theorem 7.3.1. *Let G be a reductive algebraic group and assume that $p \geq 3$ and p is a very good prime. Let $w \in W$ such that $w(\Phi_0) = \Phi_J$.*

- (a) *If $J \subseteq \Pi$ satisfies (A1), then*
  - (i) $\mathrm{H}^{2\bullet}(G_1, F) \cong \operatorname{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*)$*;*
  - (ii) $\mathrm{H}^{2\bullet+1}(G_1, F) = 0$.
- (b) *If $J \subseteq \Pi$ satisfies (A1) and (A2), then*
  - (i) $\mathrm{H}^{2\bullet}(G_1, F) \cong F[\mathcal{N}(\Phi_0)]$
  - (ii) $\mathrm{H}^{2\bullet+1}(G_1, F) = 0$.

Proof. Let w be in W such that $w(\Phi_0) = \Phi_{w\cdot 0} = \Phi_J$ where $J \subseteq \Pi$. One can use an argument similar to that given in Section 5 with the functors

$$\operatorname{Hom}_{G_1}(\mathbb{C}, \operatorname{ind}_{P_J}^G(-)) \text{ and } \operatorname{ind}_{P_J/(P_J)_1}^{G/G_1}(\operatorname{Hom}_{(P_J)_1}(\mathbb{C}, -))$$

(from P_J-mod to G/G_1-mod) to construct a first quadrant spectral sequence

$$E_2^{i,j} = [R^i \operatorname{ind}_{P_J}^G(\mathrm{H}^j((P_J)_1, \operatorname{ind}_B^{P_J} w\cdot 0)^{(-1)})]^{(1)} \Rightarrow \mathrm{H}^{i+j-\ell(w)}(G_1, F). \tag{7.3.1}$$

Next one can apply the LHS spectral sequence and the fact that $\operatorname{ind}_B^{P_J} w\cdot 0$ is an injective $(L_J)_1$-module to conclude that

$$\operatorname{Hom}_{(L_J)_1}(F, \operatorname{ind}_B^{P_J} w\cdot 0 \otimes \mathrm{H}^j((U_J)_1, F)) = \mathrm{H}^j((P_J)_1, \operatorname{ind}_B^{P_J} w\cdot 0). \tag{7.3.2}$$

In order to compute $\mathrm{H}^\bullet((U_J)_1, F)$, there is a first quadrant spectral sequence of P_J-modules [Jan1, I 9.16]:

$$E_0^{i,j} = S^i(\mathfrak{u}_J^*)^{(1)} \otimes \Lambda^j(\mathfrak{u}_J^*) \Rightarrow \mathrm{H}^{2i+j}((U_J)_1, F). \tag{7.3.3}$$

Since $\operatorname{ind}_B^{P_J} w\cdot 0$ is an injective $(L_J)_1$-module, we can compose the spectral sequence in (7.3.3) with $\operatorname{Hom}_{(L_J)_1}(F, \operatorname{ind}_B^{P_J} w\cdot 0 \otimes -)$ and use (7.3.2) to construct a spectral sequence:

$$E_0^{i,j} = S^i(\mathfrak{u}_J^*)^{(1)} \otimes \operatorname{Hom}_{(L_J)_1}(F, \operatorname{ind}_B^{P_J} w\cdot 0 \otimes \Lambda^j(\mathfrak{u}^*)) \Rightarrow \mathrm{H}^{2i+j}((P_J)_1, \operatorname{ind}_B^{P_J} w\cdot 0). \tag{7.3.4}$$

Now one can use the same analysis as in Section 4 with the Steinberg module to conclude that (analogous to Theorem 4.3.1)

$$\mathrm{Hom}_{(L_J)_1}(F, \operatorname{ind}_B^{P_J} w\cdot 0 \otimes \Lambda^j(\mathfrak{u}_J^*)) \cong \begin{cases} F & \text{if } j=\ell(w) \\ 0 & \text{else.} \end{cases}$$

Therefore, the spectral sequence (7.3.4) collapses and yields:

$$\mathrm{H}^i((P_J)_1, \operatorname{ind}_B^P w\cdot 0) \cong S^{\frac{i-\ell(w)}{2}}(\mathfrak{u}_J^*)^{(1)}.$$

By using this isomorphism and assumption (A1), the spectral sequence (7.3.1) collapses and we obtain part (a) of the theorem. For part (b), we simply use (A2) and Theorem 3.7.1. □

Once again we can consider the case not covered in the preceding theorem which happens only when Φ is of type A_n. This result encompasses Andersen and Jantzen's computation when $p=n+1$ [AJ, §6.A] (corresponding to $m=0$). The proof follows the line of reasoning given in Section 5.5 and Theorem 7.3.1. Details are left to the reader.

Theorem 7.3.2. *Let $G=\mathrm{GL}_n(F)$. Assume that $p\geq 3$ and $p \mid n+1$ with $n+1=l(m+1)$. Let $w\in W$ such that $w(\Phi_0)=\Phi_J$. If $J\subseteq \Pi$ satisfies (A1)$'$, then*

(a) $\mathrm{H}^{2\bullet}(G_1,F)\cong \bigoplus_{t=0}^{p-1} \operatorname{ind}_{P_J}^G S^{\frac{2\bullet-(m+1)t(p-t)}{2}}(\mathfrak{u}_J^*)\otimes \varpi_{t(m+1)}$;
(b) $\mathrm{H}^{2\bullet+1}(G_1,F)=0$.

7.4. When J consists of a single simple root the assumptions (A1) and (A2) were verified by Thomsen [Th, Prop. 7, proof of Thm. 2, Lemma 14], and (A2) has also been verified by Kumar, Lauritzen and Thomsen [KLT, Thm. 6]. In this case $\mathcal{N}(\Phi_0)$ is the closure of the subregular orbit $\mathcal{O}_{\text{subreg}}$ which occurs precisely when $p=h-1$. Consequently, we have the following corollary.

Corollary 7.4.1. *Let G be a reductive algebraic group and assume that $p=h-1$.*

(a) $\mathrm{H}^{2\bullet}(G_1,F)\cong F[\overline{\mathcal{O}}_{\text{subreg}}]$
(b) $\mathrm{H}^{2\bullet+1}(G_1,F)=0$.

Using Thomsen's work [Th] one can also compute the cohomology algebra for more general examples when J consists of pairwise orthogonal simple short roots. In these cases, (A1) and (A2) are satisfied. In particular, Theorem 7.3.1 can be applied to the situation when Φ is of types A_n, C_n, D_n with $p>h/2$, Φ is of type B_n with $p\geq h/2$, Φ is of type F_4 with $p\geq 7$, Φ is of type E_6 with $p\geq 7$, Φ is of type E_7 with $p\geq 11$, and Φ is of type E_8 with $p\geq 17$. Recent results of Christophersen [C, Thm. 1, Example 3.15] verify (A1) and (A2) for the group E_6 when $p\geq 5$ for those subsets $J\subseteq\Pi$ listed in Section 9.1, which gives the following result.

Corollary 7.4.2. *Let G be a reductive algebraic group with $\Phi=E_6$ and $p\geq 5$*

(a) $\mathrm{H}^{2\bullet}(G_1,F)\cong F[\mathcal{N}(\Phi_0)]$
(b) $\mathrm{H}^{2\bullet+1}(G_1,F)=0$.

8. Support varieties of induced modules/Weyl modules

8.1. In this section, we will assume that l satisfies Assumption 1.3.3. Let $R = \mathrm{H}^{2\bullet}(u_\zeta(\mathfrak{g}), \mathbb{C})$. We have proven that R is a commutative finitely generated $\mathbb{C}$-algebra. Moreover, if M is a finite dimensional $u_\zeta(\mathfrak{g})$-module, then $\mathrm{H}^{\bullet}(u_\zeta(\mathfrak{g}), M)$ is a finitely-generated R-module. Let J_M be the annhilator of the action of R on $\mathrm{H}^{\bullet}(u_\zeta(\mathfrak{g}), M \otimes M^*)$, and set $\mathcal{V}_{\mathfrak{g}}(M)$ equal to the maximum ideal spectrum of R/J_M. The variety $\mathcal{V}_{\mathfrak{g}}(M)$ is called the *support variety* of M. This variety is a conical subvariety of $\mathcal{N}(\Phi_0)$. Moreover, if M is a $U_\zeta(\mathfrak{g})$-module then $\mathcal{V}_{\mathfrak{g}}(M)$ is stable under the adjoint action of G.

8.2. Let $\lambda \in X$ and let $\Phi_\lambda = \{\alpha \in \Phi : \langle \lambda + \rho, \alpha^\vee \rangle \in l\mathbb{Z}\}$. We can now present a result which allows one to compute the supports of induced/Weyl modules provided that one can conjugate this stabilizer set into a subroot system which is generated by simple roots. Using the notation of Section 2.10, let $\mathrm{H}^0_\zeta(\lambda)$ denote the $U_\zeta(\mathfrak{g})$-module induced from the one dimensional $U_\zeta(\mathfrak{b})$-module $\mathbb{C}_\lambda$ determined by the character λ. In what follows, consider $\mathrm{H}^0_\zeta(\lambda)$ to be a $u_\zeta(\mathfrak{g})$-module by restriction.

Theorem 8.2.1. *Let $\lambda \in X_+$. Then*
(a) $\dim \mathcal{V}_{\mathfrak{g}}(H^0_\zeta(\lambda)) \geq |\Phi| - |\Phi_\lambda|$;
(b) *If there exists $w \in W$ such that $w(\Phi_\lambda) = \Phi_J$, then $\mathcal{V}_{\mathfrak{g}}(H^0_\zeta(\lambda)) = G \cdot \mathfrak{u}_J$.*

Proof. For part (a) one can use the proof given in [UGA3, §2] by replacing G_1 with $u_\zeta(\mathfrak{g})$ (resp. B_1 by $u_\zeta(\mathfrak{b})$). One can do the same in [NPV, §5 and Cor 6.1.3] to see that $\mathcal{V}_{\mathfrak{g}}(H^0_\zeta(\lambda)) \subseteq G \cdot \mathfrak{u}_J$. Part (b) now follows by using the fact that $G \cdot \mathfrak{u}_J$ is irreducible and $\dim G \cdot \mathfrak{u}_J = |\Phi| - |\Phi_\lambda|$. □

The following result (generalizing [NPV, (6.2.1) Thm.]) provides a description of the support varieties of the modules $\mathrm{H}^0_\zeta(\lambda)$ for $\lambda \in X(T)_+$ in terms of closures of Richardson orbits. This is Theorem 1.3.5. There are restrictions on l even in the case when $l > h$. See the remarks in Sections 3.1 and 8.3.

Corollary 8.2.2. *Let $\mathfrak{g}$ be a complex simple Lie algebra and l be an odd positive integer which is good for Φ. If $\lambda \in X_+$, then there exists $w \in W$ such that $w(\Phi_\lambda) = \Phi_J$ for some $J \subseteq \Pi$, and $\mathcal{V}_{\mathfrak{g}}(H^0_\zeta(\lambda)) = G \cdot \mathfrak{u}_J$.*

Proof. This follows immediately from Lemma 3.1.1 and Theorem 8.2.1. □

8.3. The statement of [Ost, Thm. 6.1] is not correct. Corollary 8.2.2 provides an accurate formulation. Even though $l > h$, one can still have Φ_λ not being W conjugate to Φ_J for some $J \subseteq \Pi$. For example, let Φ be of type F_4 and let $l = 15$ with $h = 13$. Then

$$\begin{aligned} \Phi_{4\rho} &= \{\alpha \in \Phi : \langle 4\rho + \rho, \alpha^\vee \rangle \in 15\mathbb{Z}\} \\ &= \{\alpha \in \Phi : \langle 5\rho, \alpha^\vee \rangle \in 15\mathbb{Z}\} \\ &= \{\alpha \in \Phi : \langle \rho, \alpha^\vee \rangle \in 3\mathbb{Z}\}. \end{aligned}$$

In [UGA2, §4.2], it is shown that $|\Phi_{4\rho}| = 12$. By a direct calculation with subroot systems in F_4 given by $J \subseteq \Pi$, there is no such J with $|\Phi_J| = 12$. Hence, in this case there exists no $w \in W$ with $w(\Phi_{4\rho}) = \Phi_J$ where $J \subseteq \Pi$. Furthermore, the same methods used in [UGA3, Cor. 2.5] show that $\dim \mathcal{V}_{\mathfrak{g}}(H^0_\zeta(4\rho)) \geq |\Phi| - |\Phi_{4\rho}| = 36$. On the other hand, let $I \subseteq \Pi$ and $x_I = \sum_{\alpha \in I} x_\alpha$ (i. e., the regular element in the Levi subalgebra determined by I). The arguments in [NPV, §4] [UGA3, §2.6] demonstrate that if $w(\Phi_{4\rho}) \cap \Phi_I \neq \varnothing$ for all $w \in W$ then $x_I \notin \mathcal{V}_{\mathfrak{g}}(H^0_\zeta(\lambda))$. This information in addition to the "constrictor" technique provided by [UGA3, §2.8] can be used to prove that

$$\mathcal{V}_{\mathfrak{g}}(H^0_\zeta(4\rho)) = \overline{\mathcal{O}(A_1 + \widetilde{A}_2)}$$

which is not a Richardson orbit. This is essentially the same calculation as for the support variety of the trivial module over F_4 when the characteristic of the field is three (see [UGA2, §4.2]).

9. Appendix: Tables

9.1. For each odd integer $l > 1$ which is not equal to a bad prime for Φ, the following tables give an element $w \in W$ and subset $J \subset \Pi$ such that $w(\Phi_0) = \Phi_J$. The element w is chosen to be a distinguished right W_J-coset representative, and identified by means of a reduced expression in terms of the simple reflections $s_i = s_{\alpha_i}$. Similar short-hand is used to denote the simple roots in J. Also given is the type of Φ_0 (or equivalently Φ_J), the Bala-Carter label for the nilpotent orbit $\mathcal{N}(\Phi_0) = G \cdot \mathfrak{u}_J$, and the dimension of that orbit.

Type E_6:

l	$\dim \mathcal{N}(\Phi_0)$	Φ_0	J	orbit
5	62	$A_2 \times A_1 \times A_1$	$\{1,2,3,5\}$	$A_4 + A_1$
7	66	$A_1 \times A_1 \times A_1$	$\{2,3,5\}$	$E_6(a_3)$
9	70	A_1	$\{4\}$	$E_6(a_1)$
11	70	A_1	$\{4\}$	$E_6(a_1)$
≥ 12	72	$\varnothing$	$\varnothing$	E_6

l	w
5	$s_4s_3s_2s_1s_5s_4s_2s_5s_6s_3s_5s_6s_2s_4s_1s_3s_4s_3s_2s_1s_4s_3s_1s_4s_3s_2s_1$
7	$s_4s_2s_1s_3s_4s_6s_5s_4s_3s_6s_1s_5s_3s_2s_4s_6s_5$
9	$s_3s_1s_2s_5s_4s_6s_5s_3$
11	$s_5s_6s_2s_3s_4s_5s_1s_3s_4s_2$

Type E_7:

l	$\dim \mathcal{N}_1(\Phi_0)$	Φ_0	J	orbit
5	106	$A_3 \times A_2 \times A_1$	$\{1,2,3,5,6,7\}$	$A_4 + A_2$
7	114	$A_2 \times A_1 \times A_1 \times A_1$	$\{1,2,3,5,7\}$	A_6
9	118	$A_1 \times A_1 \times A_1 \times A_1$	$\{2,3,5,7\}$	$E_6(a_1)$
11	120	$A_1 \times A_1 \times A_1$	$\{2,3,5\}$	$E_7(a_3)$
13	122	$A_1 \times A_1$	$\{4,6\}$	$E_7(a_2)$
15	124	A_1	$\{1\}$	$E_7(a_1)$
17	124	A_1	$\{1\}$	$E_7(a_1)$
≥ 18	126	$\varnothing$	$\varnothing$	E_7

l	w
5	$s_4s_5s_3s_2s_4s_2s_3s_1s_3s_6s_5s_4s_3s_5s_6s_5s_3s_2s_4s_2s_3s_2s_4s_6s_5s_4s_7s_6s_4s_3s_5s_6$
7	$s_4s_2s_3s_6s_5s_4s_6s_3s_1s_2s_5s_4s_3s_7s_6s_5s_4s_2$
9	$s_4s_2s_5s_4s_1s_6s_5s_3s_4s_5s_2s_3s_4s_7s_6s_1s_2s_3s_4s_5s_6s_7$
11	$s_4s_3s_2s_5s_6s_5s_7s_6s_1s_3s_4s_2s_3s_5s_4s_3s_1s_2s_6s_5s_4s_3$
13	$s_7s_5s_6s_2s_3s_4s_5s_1s_2s_3s_4s_5s_6s_7s_1s_2s_3s_4s_5s_6$
15	$s_3s_4s_2s_5s_4s_3s_6s_5s_4s_2s_7s_6s_5s_4$
17	$s_3s_4s_2s_5s_4s_3s_6s_5s_4s_7s_6s_5s_2s_4s_3s_1$

Type E_8:

l	$\dim \mathcal{N}(\Phi_0)$	Φ_0	J	orbit
7	212	$A_4 \times A_2 \times A_1$	$\{1,2,3,5,6,7,8\}$	$A_6 + A_1$
9	220	$A_3 \times A_2 \times A_1$	$\{1,2,4,6,7,8\}$	$E_8(b_6)$
11	224	$A_2 \times A_2 \times A_1 \times A_1$	$\{1,2,3,5,7,8\}$	$E_8(a_6)$
13	228	$A_2 \times A_1 \times A_1 \times A_1$	$\{2,3,5,6,8\}$	$E_8(a_5)$
15	232	$A_1 \times A_1 \times A_1 \times A_1$	$\{1,4,6,8\}$	$E_8(a_4)$
17	232	$A_1 \times A_1 \times A_1 \times A_1$	$\{2,3,5,7\}$	$E_8(a_4)$
19	234	$A_1 \times A_1 \times A_1$	$\{2,3,5\}$	$E_8(a_3)$
21	236	$A_1 \times A_1$	$\{6,8\}$	$E_8(a_2)$
23	236	$A_1 \times A_1$	$\{6,8\}$	$E_8(a_2)$
25	238	A_1	$\{1\}$	$E_8(a_1)$
27	238	A_1	$\{1\}$	$E_8(a_1)$
29	238	A_1	$\{1\}$	$E_8(a_1)$
≥ 30	240	$\varnothing$	$\varnothing$	E_8

l	w
7	$s_4s_3s_2s_4s_5s_4s_2s_1s_3s_1s_4s_2s_3s_5s_4s_6s_5s_4s_3s_1s_2s_4s_3s_7s_6s_8s_7s_5$
–	$s_4s_2s_6s_5s_4s_3s_4s_5s_6s_7s_6s_8s_7s_8s_1s_3s_2s_4s_5s_6s_4s_1s_3s_2s_4s_1s_2s_3$
9	$s_5s_6s_4s_5s_2s_4s_3s_4s_5s_6s_1s_2s_3s_4s_2s_5s_4s_3s_4s_2s_6s_5s_4s_7s_6s_5s_8s_7s_6s_1s_3$
–	$s_4s_2s_5s_4s_3s_4s_5s_6s_5s_4s_1s_2s_3s_4s_5s_1s_3s_4s_7s_6s_5s_2s_4s_5s_6s_7s_1s_2s_3s_4s_5$
11	$s_4s_2s_5s_3s_4s_3s_2s_6s_5s_4s_2s_7s_6s_3s_5s_4s_5s_6s_7s_1s_3s_4s_5s_2s_4s_6s_5s_8$
–	$s_7s_6s_7s_1s_3s_4s_5s_2s_4s_2s_6s_7s_3s_1s_3s_4s_2s_5s_4s_3s_4s_3s_1s_3s_8s_7s_6s_5$
13	$s_4s_2s_3s_4s_5s_7s_6s_8s_7s_5s_4s_2s_3s_4s_1s_3s_4s_2s_5s_4s_3s_1s_2s_6s_5$
–	$s_4s_3s_2s_5s_4s_3s_1s_7s_8s_6s_5s_7s_6s_4s_2s_5s_3s_4s_5s_2s_6s_3s_1s_4s_3$
15	$s_3s_4s_2s_5s_4s_3s_6s_5s_4s_7s_8s_6s_5s_7s_6s_2s_4s_5s_2s_4s_6s_7s_5s_6s_1s_3s_8s_7$
–	$s_4s_5s_3s_4s_1s_3s_6s_5s_4s_2s_7s_6s_5s_8s_3s_4s_2s_5s_4s_3s_6s_5s_4s_2s_7s_6s_5s_4$
17	$s_4s_2s_5s_4s_6s_5s_7s_6s_1s_3s_4s_2s_5s_4s_8s_7s_6s_5s_3s_4s_2s_1s_3s_4s_5s_6s_7s_8s_4s_5s_6s_7s_2s_4s_5s_6s_1s_3s_4$
–	$s_2s_5s_4s_3s_1s_8s_7s_4s_5s_3s_4s_1s_3s_6s_5s_4s_2s_7s_6s_5s_4s_8s_7s_3s_4s_2s_5s_4s_3s_6s_5s_4s_2s_7s_6s_5s_4s_3s_1$
19	$s_4s_2s_3s_4s_1s_3s_6s_5s_4s_2s_7s_6s_5s_4s_3s_1s_8s_7s_6s_5s_4s_3s_2s_4s_5s_6s_7s_8s_4s_5s_3s_4$
–	$s_1s_3s_6s_5s_4s_2s_7s_6s_5s_4s_8s_7s_6s_5s_3s_4s_2s_5s_4s_3s_6s_5s_4s_2s_7s_6s_5s_4s_3s_1s_8s_7$
21	$s_7s_6s_5s_4s_2s_3s_4s_5s_6s_7s_1s_3s_4s_5s_6s_2s_4s_5s_3s_4s_1s_3s_2s_4s_5s_6$
–	$s_3s_4s_2s_5s_4s_3s_6s_5s_4s_2s_7s_6s_5s_4s_3s_1s_8s_7s_6s_5$
23	$s_7s_6s_5s_4s_2s_3s_4s_5s_6s_7s_1s_3s_4s_5s_6s_2s_4s_5s_3s_4s_1s_3s_2s_4s_5s_6s_7s_8$
–	$s_3s_4s_2s_5s_4s_3s_6s_5s_4s_2s_7s_6s_5s_4s_3s_1s_8s_7s_6s_5s_4s_2$
25	$s_3s_4s_2s_5s_4s_3s_6s_5s_4s_2s_7s_6s_5s_4s_3s_1s_8s_7s_6s_5s_4s_2s_3s_4$
27	$s_3s_4s_2s_5s_4s_3s_6s_5s_4s_2s_7s_6s_5s_4s_3s_1s_8s_7s_6s_5s_4s_2s_3s_4s_5s_6$
29	$s_3s_4s_2s_5s_4s_3s_6s_5s_4s_2s_7s_6s_5s_4s_3s_1s_8s_7s_6s_5s_4s_2s_3s_4s_5s_6s_7s_8$

Type F_4:

l	$\dim \mathcal{N}(\Phi_0)$	Φ_0	w	J	orbit
5	40	$A_2 \times A_1$	$s_2s_3s_4s_2s_3s_2s_1s_2s_3s_1s_2s_3$	$\{1,3,4\}$	$F_4(a_3)$
7	44	$A_1 \times A_1$	$s_2s_1s_4s_3s_2s_3s_4s_1s_3s_2s_4s_3s_2$	$\{1,3\}$	$F_4(a_2)$
9	46	A_1	$s_4s_2s_3s_1s_2$	$\{3\}$	$F_4(a_1)$
11	46	A_1	$s_4s_2s_3s_1s_2s_3s_4$	$\{3\}$	$F_4(a_1)$
≥ 12	48	$\varnothing$	–	$\varnothing$	F_4

Type G_2:

l	$\dim \mathcal{N}_1(\Phi_0)$	Φ_0	w	J	orbit
5	10	A_1	s_2s_1	$\{1\}$	$G_2(a_1)$
≥ 6	12	$\varnothing$	–	$\varnothing$	G_2

9.2. The information listed below for the exceptional Lie algebras is used for justifying the arguments given in Section 4.9. For each l, the element $w \in W$ referred to here is the w identified in the tables of Section 9.1. All weights (e.g., $w \cdot 0$) are listed in the weight (or ϖ) basis.

Type E_6:

l	J	$w \cdot 0$	$-w_{0,J}(w \cdot 0)$
11	$\{4\}$	$(0,-6,-6,10,-6,0)$	$(0,-4,-4,10,-4,0)$
9	$\{4\}$	$(1,-5,-5,8,-5,1)$	$(-1,-3,-3,8,-3,-1)$
7	$\{2,3,5\}$	$(-3,6,6,-11,6,-6)$	$(-3,6,6,-7,6,0)$
5	$\{1,2,3,5\}$	$(4,4,4,-10,4,-5)$	$(4,4,4,-6,4,1)$

l	$\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$	λ	$\langle \lambda, \delta^\vee \rangle$
11	10	$(0,-4,-4,10,-4,0)$	10
9	8	$(0,-4,-4,10,-4,0)$	10
7	18	$(-2,6,6,-7,6,-2)$	18
5	16	$(3,6,2,-6,6,-2)$	17

l	α	$\langle x, \alpha^\vee \rangle$	$\langle -w_{0,J}(w \cdot 0), \alpha^\vee \rangle$
7	α_1	$-4 \le * \le 1$	-3
7	α_4	$-7 \le * \le -6$	-7
7	α_6	$-4 \le * \le 1$	0

Type E_7:

l	J	$w \cdot 0$	$-w_{0,J}(w \cdot 0)$
17	$\{1\}$	$(16,0,-11,0,0,0,0)$	$(16,0,-5,0,0,0,0)$
15	$\{1\}$	$(14,0,-10,0,0,1,0)$	$(14,0,-4,0,0,0,0)$
13	$\{4,6\}$	$(0,-7,-7,12,-13,12,-8)$	$(0,-5,-5,12,-11,12,-4)$
11	$\{2,3,5\}$	$(-6,10,10,-17,10,-6,0)$	$(-4,10,10,-13,10,-4,0)$
9	$\{2,3,5,7\}$	$(-5,8,8,-14,8,-8,8)$	$(-3,8,8,-10,8,-8,8)$
7	$\{1,2,3,5,7\}$	$(6,6,6,-14,6,-7,6)$	$(6,6,6,-10,6,-5,6)$
5	$\{1,2,3,5,6,7\}$	$(4,4,4,-15,4,4,4)$	$(4,4,4,-9,4,4,4)$

l	$\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$	λ	$\langle \lambda, \delta^\vee \rangle$
17	16	$(16,0,-5,0,0,0,0,)$	16
15	14	$(16,0,-5,0,0,0,0,)$	16
13	24	$(0,-5,-5,12,-11,12,-4)$	24
11	30	$(-4,10,10,-13,10,-4)$	30
9	32	$(-4,8,10,-11,8,-7,8)$	34
7	30	$(5,8,4,-10,8,-7,8)$	33
5	24	$(5,10,4,-10,3,0,5)$	27

l	α	$\langle x, \alpha^\vee \rangle$	$\langle -w_{0,J}(w \cdot 0), \alpha^\vee \rangle$
15	$\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7$	$-10 \le * \le 12$	0
15	α_3	$-15 \le * \le 4$	-4
9	α_1	$-8 \le * \le 4$	-3
9	α_4	$-13 \le * \le -6$	-10
9	α_6	$-12 \le * \le 0$	-8

Type E_8:

l	J	$w \cdot 0$	$-w_{0,J}(w \cdot 0)$
29	$\{1\}$	$(28, 0, -17, 0, 0, 0, 0, 0)$	$(28, 0, -11, 0, 0, 0, 0, 0)$
27	$\{1\}$	$(26, 0, -17, 1, 0, 0, 0, 0)$	$(26, 0, -9, -1, 0, 0, 0, 0)$
25	$\{1\}$	$(24, 0, -15, 0, 0, 1, 0, 0)$	$(24, 0, -9, 0, 0, -1, 0, 0)$
23	$\{6, 8\}$	$(0, 0, 0, 0, -13, 22, -25, 22)$	$(0, 0, 0, 0, -9, 22, -19, 22)$
21	$\{6, 8\}$	$(2, 0, 0, 0, -13, 20, -21, 20)$	$(-2, 0, 0, 0, -7, 20, -19, 20)$
19	$\{2, 3, 5\}$	$(-10, 18, 18, -29, 18, -13, 4, 0)$	$(-8, 18, 18, -25, 18, -5, -4, 0)$
17	$\{2, 3, 5, 7\}$	$(-14, 16, 16, -25, 16, -16, 16, -11)$	$(-2, 16, 16, -23, 16, -16, 16, -5)$
15	$\{1, 4, 6, 8\}$	$(14, -10, -17, 14, -13, 14, -13, 14)$	$(14, -4, -11, 14, -15, 14, -15, 14)$
13	$\{2, 3, 5, 6, 8\}$	$(-7, 12, 12, -26, 12, 12, -21, 12)$	$(-5, 12, 12, -22, 12, 12, -15, 12)$
11	$\{1, 2, 3, 5, 7, 8\}$	$(10, 10, 10, -22, 10, -18, 10, 10)$	$(10, 10, 10, -18, 10, -12, 10, 10)$
9	$\{1, 2, 4, 6, 7, 8\}$	$(8, 8, -13, 8, -24, 8, 8, 8)$	$(8, 8, -11, 8, -16, 8, 8, 8)$
7	$\{1, 2, 3, 5, 6, 7, 8\}$	$(6, 6, 6, -25, 6, 6, 6, 6)$	$(6, 6, 6, -17, 6, 6, 6, 6)$

l	$\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$	λ	$\langle \lambda, \delta^\vee \rangle$
29	28	$(28, 0, -11, 0, 0, 0, 0, 0)$	28
27	26	$(28, 0, -11, 0, 0, 0, 0, 0)$	28
25	24	$(28, 0, -11, 0, 0, 0, 0, 0)$	28
23	44	$(0, 0, 0, 0, -8, 22, -21, 22)$	44
21	40	$(0, 0, 0, 0, -8, 22, -21, 22)$	44
19	54	$(-8, 18, 18, -25, 18, -8, 0, 0)$	54
17	64	$(-7, 16, 16, -22, 16, -15, 16, -6)$	64
15	56	$(16, -5, -15, 16, -15, 16, -15, 16)$	64
13	60	$(-7, 16, 16, -21, 7, 8, -14, 16)$	63
11	60	$(8, 16, 7, -21, 16, -14, 8, 7)$	62
9	48	$(18, 7, -16, 10, -14, 6, 0, 10)$	51
7	42	$(9, 18, 8, -21, 6, 0, 0, 9)$	50

l	α	$\langle x, \alpha^\vee \rangle$	$\langle -w_{0,J}(w \cdot 0), \alpha^\vee \rangle$
27	$\alpha_2, \alpha_5, \alpha_6, \alpha_7, \alpha_8$	$-18 \leq * \leq 20$	0
27	α_3	$-27 \leq * \leq 4$	-9
27	α_4	$-18 \leq * \leq 20$	-1
25	$\alpha_2, \alpha_4, \alpha_5, \alpha_7, \alpha_8$	$-20 \leq * \leq 22$	0
25	α_3	$-27 \leq * \leq 6$	-9
25	α_6	$-20 \leq * \leq 22$	-1
23	$\alpha_1, \alpha_2, \alpha_3, \alpha_4$	$-10 \leq * \leq 12$	0
23	α_5	$-20 \leq * \leq 1$	-9
23	α_7	$-26 \leq * \leq -14$	-19
21	α_1	$-14 \leq * \leq 16$	-2
21	$\alpha_2, \alpha_3, \alpha_4$	$-14 \leq * \leq 16$	0
21	α_5	$-24 \leq * \leq 6$	-7
21	α_7	$-26 \leq * \leq -9$	-19
19	α_1	$-16 \leq * \leq 1$	-8
19	α_4	$-25 \leq * \leq -24$	-25
19	α_6	$-16 \leq * \leq 1$	-5
19	α_7	$-10 \leq * \leq 12$	-4
19	α_8	$-10 \leq * \leq 12$	0
17	α_1	$-14 \leq * \leq 1$	-2
17	α_4	$-25 \leq * \leq -18$	-23
17	α_6	$-20 \leq * \leq -8$	-16
17	α_8	$-14 \leq * \leq 1$	-5
13	α_1	$-17 \leq * \leq 5$	-5
13	α_4	$-24 \leq * \leq -13$	-22
13	α_7	$-22 \leq * \leq -3$	-15
11	α_4	$-24 \leq * \leq -15$	-18
11	α_6	$-21 \leq * \leq -5$	-12

Type F_4:

l	J	$w \cdot 0$	$-w_{0,J}(w \cdot 0)$	$\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$	λ	$\langle \lambda, \delta^\vee \rangle$
11	$\{3\}$	$(0,-6,10,-6)$	$(0,-4,10,-4)$	10	$(0,-4,10,-4)$	10
9	$\{3\}$	$(1,-5,8,-5)$	$(-1,-3,8,-3)$	8	$(0,-4,10,-4)$	10
7	$\{1,3\}$	$(6,-7,6,-6)$	$(6,-5,6,0)$	12	$(4,-5,9,-3)$	13
5	$\{1,3,4\}$	$(4,-8,4,4)$	$(4,-4,4,4)$	12	$(4,-4,4,4)$	12

Type G_2:

l	J	$w \cdot 0$	$-w_{0,J}(w \cdot 0)$	$\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$	λ	$\langle \lambda, \delta^\vee \rangle$
5	$\{1\}$	$(4,-3)$	$(4,-1)$	4	$(4,-1)$	4

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